Section 2: Applications of de Moivre's theorem

Exercise level 3

1. If $p\cos 5\theta + q\cos 3\theta + r\cos\theta \equiv \cos^5\theta + \cos^3\theta + \cos\theta$, find *p*, *q* and *r*.

2. Show that $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$, where $z = \cos\theta + \mathrm{i}\sin\theta$, and find the corresponding identity for $\sin n\theta$. Hence show that $\cos n\theta \times \sin m\theta = \tfrac{1}{2}\sin(n+m)\theta + \tfrac{1}{2}\sin(m-n)\theta$.

3. Find $\displaystyle\sum_{k=0}^{n} {}^nC_r \cos(a+bk)$.

4. Find all the solutions to $\mathrm{e}^z = 1 - \mathrm{i}$ in the form $z = x + \mathrm{i}y$.

5. (i) Plot the solutions to $\mathrm{e}^z = \mathrm{e}^{3+\mathrm{i}}$ that are near the origin.
   (ii) Plot the solutions to $|z| = 3$ on your diagram.
   (iii) Research the difference between an algebraic and a transcendental number.
   (iv) Let S = {solutions to $\mathrm{e}^z = \mathrm{e}^{p+q\mathrm{i}}$, $p, q \in \mathbb{Z}$}
   and let R = {solutions to $|z| = k$, $k \in \mathbb{N}$ }
   Show that the only points *z* in both S and R have Im(*z*) $\in \mathbb{Z}$.
   (e.g. $\mathrm{e}^z = \mathrm{e}^{3+4\mathrm{i}}$ has a solution lying on $|z| = 5$).

6. In this question you will look at solving the equation $\mathrm{e}^z = z^{\mathrm{e}}$. (Note: *z* = e is one root).
   (i) Show that if $z = x + \mathrm{i}y$, then the LHS of the equation is $\mathrm{e}^x\mathrm{e}^{\mathrm{i}y}$.
   (ii) If $z = r\mathrm{e}^{\mathrm{i}\theta}$, show that the RHS of the equation is $r^{\mathrm{e}}\mathrm{e}^{\mathrm{i}\theta\mathrm{e}}$.
   (iii) By putting $x = r\cos\theta, y = r\sin\theta$ and eliminating *r*, show that θ satisfies the equation
   $$\mathrm{e}^{(\mathrm{e}\theta - 2n\pi)\cot\theta} - \left(\frac{\mathrm{e}\theta - 2n\pi}{\sin\theta}\right)^{\mathrm{e}} = 0.$$
   (iv) By using graphing software, find the solution to $\mathrm{e}^z = z^{\mathrm{e}}$ when *n* = 1.

7. (i) Show $1 - \mathrm{e}^{\frac{2n\pi}{3}\mathrm{i}} = -2\mathrm{i}\mathrm{e}^{\frac{n\pi}{3}\mathrm{i}}\sin\dfrac{n\pi}{3}$ and $1 + \mathrm{e}^{\frac{2n\pi}{3}\mathrm{i}} = 2\mathrm{e}^{\frac{n\pi}{3}\mathrm{i}}\cos\dfrac{n\pi}{3}$.
   (ii) Solve $(z+1)^3 = (z-1)^3$ by multiplying out the brackets.
   (iii) Solve $(z+1)^3 = (z-1)^3$ by considering $\left(\dfrac{z+1}{z-1}\right)^3 = 1$.
   Compare your answers.

8. (i) Show that $1+e^{i\theta}=2\cos\frac{\theta}{2}e^{\frac{i\theta}{2}}$ and $-1+e^{i\theta}=2i\sin\frac{\theta}{2}e^{\frac{i\theta}{2}}$.

   (ii) The complex numbers 0, 1, z_1 and z_2 form a rhombus. Prove using geometrical complex number methods, that α is a right angle,

   (iii) Prove using geometrical complex number methods that the angle in a semicircle is 90°.