Edexcel Further Mathematics Complex numbers

Section 2: Applications of de Moivre's theorem

Exercise level 2

1. (i) Find $\cos 3\theta$ and $\sin 3\theta$ as polynomials in $\cos\theta$ and $\sin\theta$ respectively.

   (ii) Using $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$, write $\tan 3\theta$ in terms of $\tan\theta$.

2. Show that $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$ and $\sin n\theta = \dfrac{z^n - z^{-n}}{2\text{i}}$, where $z = \cos\theta + \text{i}\sin\theta$.

   Hence deduce that $2\cos(n\theta)\sin(n\theta) = \sin(2n\theta)$.

3. Express sin 5θ in terms of powers of sin θ.

4. Express $\cos^5\theta\sin^2\theta$ in terms of cosines of multiples of θ.

5. (i) Express $\text{e}^{\text{i}k\theta}$ and $\text{e}^{-\text{i}k\theta}$ in the form a + ib, and show that

   $$\text{e}^{2\text{i}\theta} - 1 = 2\text{i}\text{e}^{\text{i}\theta}\sin\theta.$$

   Series C and S are defined by

   $$C = \cos\theta + \cos 3\theta + \cos 5\theta + ... + \cos(2n-1)\theta$$

   $$S = \sin\theta + \sin 3\theta + \sin 5\theta + ... + \sin(2n-1)\theta$$

   where n is a positive integer and $0 < \theta < \dfrac{\pi}{n}$.

   (ii) Show that C + iS is a geometric series, and write down the sum of this series.

   (iii) Show that $|C + \text{i}S| = \dfrac{\sin n\theta}{\sin\theta}$, and find $\arg(C + \text{i}S)$.

   (iv) Find C and S.

6. In this question, k and θ are real numbers with 0 < k < 1 and $0 < \theta < \frac{1}{2}\pi$.

   (i) Express $(1 - k\text{e}^{\text{i}\theta})(1 - k\text{e}^{-\text{i}\theta})$ in trigonometric form.

   Infinite series C and S are defined by

   $$C = k\cos\theta + k^2\cos 2\theta + k^3\cos 3\theta + ...$$

   $$S = k\sin\theta + k^2\sin 2\theta + k^3\sin 3\theta + ...$$

   (ii) Show that C + iS is an infinite geometric series.

   (iii) By finding the sum of this series, show that $C = \dfrac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}$ and find a similar expression for S.

   (iv) Given that C = 0, show that $S = \cot\theta$.

7. (i) Express in a simplified trigonometric form

   (a) $\text{e}^{\text{i}\theta} + \text{e}^{-\text{i}\theta}$ (b) $(1 - 3\text{e}^{\text{i}\theta})(1 - 3\text{e}^{-\text{i}\theta})$

Edexcel FM Complex numbers 2 Exercise

(ii) Series C and S are defined by

$$C = \cos\theta + 3\cos 2\theta + 9\cos 3\theta + ... + 3^{n-1}\cos n\theta$$

$$S = \sin\theta + 3\sin 2\theta + 9\sin 3\theta + ... + 3^{n-1}\sin n\theta$$

Show that $C = \dfrac{\cos\theta + 3^{n+1}\cos n\theta - 3^n\cos(n+1)\theta - 3}{10 - 6\cos\theta}$

and find a similar expression for S.

8. (i) Show that $\left(1 - \dfrac{e^{i\theta}}{2}\right)\left(1 - \dfrac{e^{-i\theta}}{2}\right) = \dfrac{5}{4} - \cos\theta$

(ii) If $C = \cos\theta + \dfrac{\cos 2\theta}{2} + \dfrac{\cos 3\theta}{4} + ...$

and $S = \sin\theta + \dfrac{\sin 2\theta}{2} + \dfrac{\sin 3\theta}{4} + ...$

where these are infinite series, find C and S by considering C + iS.

9. (i) Show that $1 + e^{3i\theta} = 2\cos\left(\dfrac{3\theta}{2}\right)e^{\frac{3}{2}i\theta}$.

(ii) Find $C = 1 + \binom{n}{1}\cos 3\theta + \binom{n}{2}\cos 6\theta + ... + \binom{n}{n}\cos 3n\theta$

and the corresponding sum for $\sin\theta$.

10. (i) Write $z = 1 + \sqrt{3}i$ in the form $re^{i\theta}$.

(ii) Find the 5th roots of z in the form $re^{i\theta}$. Show these on an Argand diagram, along with z.

(iii) Two of these roots, z_1 and z_2, appear in the same quadrant. Let $w = \dfrac{z_1 + z_2}{2}$.

What is the smallest value of n so that w^n is real?

(You are given that $1 + e^{2i\theta} = 2e^{i\theta}\cos\theta$).