Section 1: de Moivre's theorem

Notes and Examples

These notes contain subsections on:

de Moivre's theorem

de Moivre's Theorem states that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta \text{ for any integer } n.$$

For positive integers this is very obvious from the rules you know for multiplying complex numbers in modulus-argument form ("add arguments, multiply moduli").

However, this is not a formal proof. de Moivre's theorem can be proved by induction for positive integers. This can then be extended to negative integers.

Using de Moivre's theorem to find powers of complex numbers

The simplest application of de Moivre's theorem is to evaluate powers of complex numbers very quickly. Notice that, even though de Moivre's theorem itself refers to a complex number with modulus 1, it can be used for any complex number because, when expressed in modulus-argument form, a complex number is written as a real number multiplied by a complex number with modulus 1. In modulus-argument form a complex number is written as follows:

$$z = x + y\mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta) \text{ where } r = |z| \text{ and } \theta = \arg(z).$$

so that

$$\begin{aligned} z^n &= (x + y\mathrm{i})^n = (r(\cos\theta + \mathrm{i}\sin\theta))^n \\ &= r^n(\cos\theta + \mathrm{i}\sin\theta)^n \\ &= r^n(\cos n\theta + \mathrm{i}\sin n\theta) \end{aligned}$$

This is illustrated in the examples below.

Example 1

Find the value of $\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^9$.

Solution

Using de Moivre's theorem with $n = 9$, $\theta = \dfrac{\pi}{3}$,

$$\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^9 = \cos\left(9\times\frac{\pi}{3}\right) + \mathrm{i}\sin\left(9\times\frac{\pi}{3}\right)$$
$$= \cos 3\pi + i\sin 3\pi$$
$$= -1$$

$\sin 3\pi = 0$ and $\cos 3\pi = -1$

de Moivre's theorem can only be used for complex numbers written in modulus-argument form. Complex numbers given in rectangular form must first be converted into modulus-argument form.

Example 2

Evaluate $(1+\mathrm{i})^{12}$.

Solution

1 + i is in the first quadrant

$|1+\mathrm{i}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg(1+\mathrm{i}) = \arctan 1 = \dfrac{\pi}{4}$

So in polar form: $\quad 1+\mathrm{i} = \sqrt{2}\left(\cos\dfrac{\pi}{4} + \mathrm{i}\sin\dfrac{\pi}{4}\right)$

Therefore:
$$(1+\mathrm{j})^{12} = \left(\sqrt{2}\right)^{12}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)^{12}.$$
$$= 64\left(\cos 3\pi + \mathrm{i}\sin 3\pi\right)$$
$$= 64\left(\cos\pi + \mathrm{i}\sin\pi\right)$$
$$= 64\times -1$$
$$= -64$$

Roots of unity

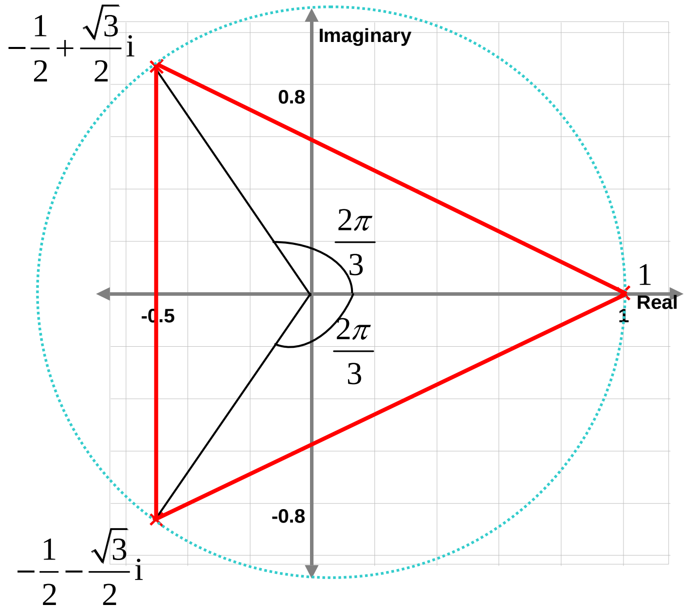
First, think about the cube roots of unity. These are numbers such that $z^3 = 1$ or equivalently such that $z^3 - 1 = 0$. Cast your mind back to the fundamental theorem of algebra: it tells you that the equation $z^3 - 1 = 0$ has three roots. One is obvious, it's 1. This means that $z-1$ is a factor of z^3-1. Dividing z^3-1 by $z-1$ gives

$$z^3 - 1 = (z-1)(z^2+z+1).$$

You can now find the other two roots of $z^3 - 1 = 0$ by applying the formula for roots of a quadratic to $z^2 + z + 1 = 0$, as follows:

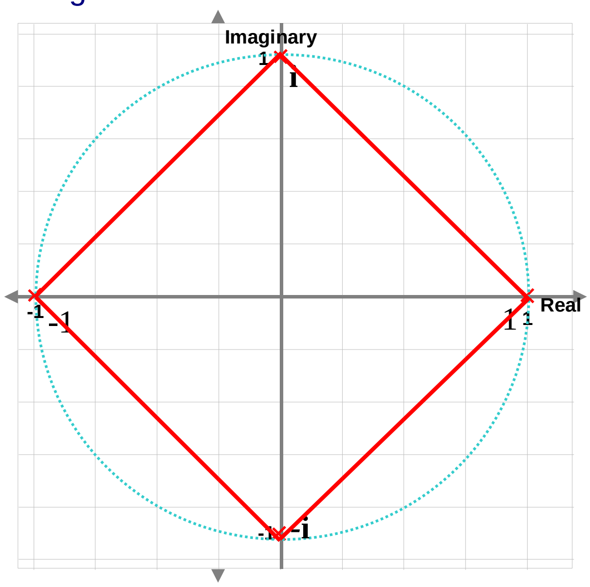
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\text{i}.$$

The three roots of $z^3 - 1 = 0$ can be plotted on an Argand diagram.

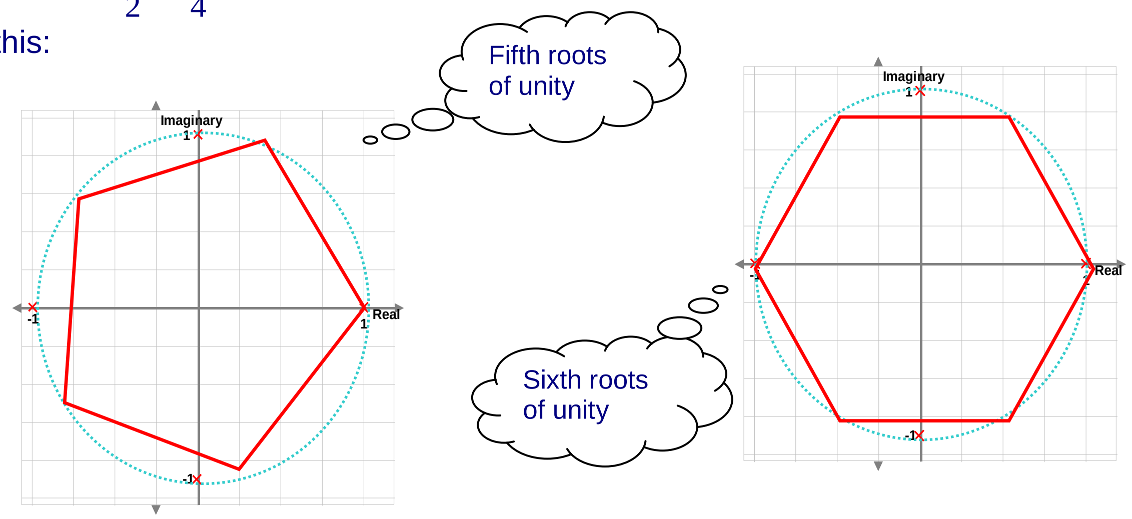


The argument of $-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}$ is $\frac{2\pi}{3}$ and the argument of $-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}$ is $-\frac{2\pi}{3}$. Notice that all the roots have modulus 1 and if you rotate any of the roots through an angle of $\frac{2\pi}{3}$ you obtain another one. Maybe you could have guessed that these would be the three roots beforehand, bearing in mind de Moivre's Theorem.

This whole process can be repeated for fourth roots of unity as $z^4 - 1 = (z-1)(z+1)(z^2+1)$. The two complex fourth roots of unity are the roots of the quadratic which appears in this factorisation. Of course, the roots of $z^2 + 1 = 0$ are $\pm\text{i}$ so that the four fourth roots of unity are $1, -1, \text{i}, -\text{i}$. These four roots of $z^4 - 1 = 0$ can be plotted on an Argand diagram.



Notice again that all the roots have modulus 1 and if you rotate a root through an angle of $\frac{\pi}{2} = \frac{2\pi}{4}$ you obtain another one. The fifth and sixth roots of unity look like this:



In summary, the nth roots of unity can be expressed as

$$z = \cos\frac{2k\pi}{n} + \mathrm{i}\sin\frac{2k\pi}{n} \qquad \text{for } k = 0, 1, 2, \ldots, n-1$$

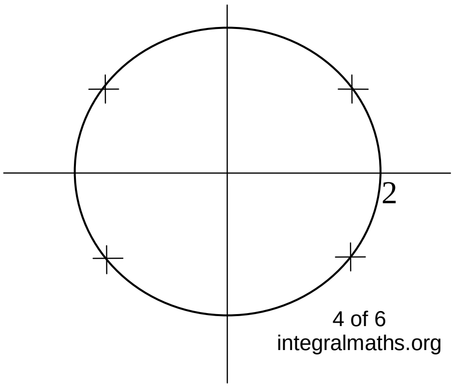
and the sum of all the nth roots of unity is always zero (make sure that you understand why this is the case.

General n^{th} roots

To find the nth root of any number, find one root and the others are placed symmetrically around a circle, centre 0, in the Argand diagram.

Example 3
Find the four roots of $z^4 = -16$.

Solution
-16 lies on the negative real axis, so has argument π.
$z^4 = -16 = 16(\cos\pi + \mathrm{i}\sin\pi)$
One root is $z = 2(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4})$
On an Argand diagram, this point lies on a circle of radius 2.
$z^4 = -16$ has four roots.
These are symmetrically placed round the circle of radius 2.



Thus the other three roots have arguments: $\frac{3\pi}{4}$, $-\frac{\pi}{4}$ and $-\frac{3\pi}{4}$.

Remember that θ must lie between $-\pi$ and π

So the roots are

$$2(\cos\tfrac{\pi}{4} + \mathrm{i}\sin\tfrac{\pi}{4}) = 2\left(\tfrac{1}{\sqrt{2}} + \mathrm{i}\tfrac{1}{\sqrt{2}}\right)$$
$$2(\cos\tfrac{3\pi}{4} + \mathrm{i}\sin\tfrac{3\pi}{4}) = 2\left(-\tfrac{1}{\sqrt{2}} + \mathrm{i}\tfrac{1}{\sqrt{2}}\right)$$
$$2(\cos(-\tfrac{\pi}{4}) + \mathrm{i}\sin(-\tfrac{\pi}{4})) = 2\left(\tfrac{1}{\sqrt{2}} - \mathrm{i}\tfrac{1}{\sqrt{2}}\right)$$
$$2(\cos(-\tfrac{3\pi}{4}) + \mathrm{i}\sin(-\tfrac{3\pi}{4})) = 2\left(-\tfrac{1}{\sqrt{2}} - \mathrm{i}\tfrac{1}{\sqrt{2}}\right)$$

Written in the $x + \mathrm{i}y$ form, these four roots are $\pm\sqrt{2}(1 \pm \mathrm{i})$

Example 4

Find the six roots of $(z-2)^6 = -27\mathrm{i}$

Solve the equation for $z-2$ and then find z.

Solution

$27\mathrm{i}$ has modulus 27 and argument $\frac{3\pi}{2}$

$\therefore$ One root for $z-2$ has modulus $(27)^{\frac{1}{6}} = \sqrt{3}$ and argument $= \frac{1}{6} \times \frac{3\pi}{2} = \frac{\pi}{4}$

On an Argand diagram, this point lies on a circle of radius $\sqrt{3}$.

$(z-2)^6 = -27\mathrm{i}$ has six roots.

These are symmetrically placed round the circle of radius $\sqrt{3}$.

Thus they are:

Remember that θ must lie between $-\pi$ and π

$$z - 2 = \sqrt{3}\left(\cos(-\tfrac{\pi}{12}) + \mathrm{i}\sin(-\tfrac{\pi}{12})\right)$$
$$\sqrt{3}\left(\cos(\tfrac{\pi}{4}) + \mathrm{i}\sin(\tfrac{\pi}{4})\right)$$
$$\sqrt{3}\left(\cos(\tfrac{7\pi}{12}) + \mathrm{i}\sin(\tfrac{7\pi}{12})\right)$$
$$\sqrt{3}\left(\cos(\tfrac{11\pi}{12}) + \mathrm{i}\sin(\tfrac{11\pi}{12})\right)$$
$$\sqrt{3}\left(\cos(-\tfrac{5\pi}{12}) + \mathrm{i}\sin(-\tfrac{5\pi}{12})\right)$$
$$\sqrt{3}\left(\cos(-\tfrac{3\pi}{4}) + \mathrm{i}\sin(-\tfrac{3\pi}{4})\right)$$

Hence the six roots for z are

$$2 + \sqrt{3}\cos(-\tfrac{\pi}{12}) + \mathrm{i}\sqrt{3}\sin(-\tfrac{\pi}{12})$$
$$2 + \sqrt{3}\cos(\tfrac{\pi}{4}) + \mathrm{i}\sqrt{3}\sin(\tfrac{\pi}{4})$$
$$2 + \sqrt{3}\cos(\tfrac{7\pi}{12}) + \mathrm{i}\sqrt{3}\sin(\tfrac{7\pi}{12})$$
$$2 + \sqrt{3}\cos(\tfrac{11\pi}{12}) + \mathrm{i}\sqrt{3}\sin(\tfrac{11\pi}{12})$$
$$2 + \sqrt{3}\cos(-\tfrac{5\pi}{12}) + \mathrm{i}\sqrt{3}\sin(-\tfrac{5\pi}{12})$$
$$2 + \sqrt{3}\cos(-\tfrac{3\pi}{4}) + \mathrm{i}\sqrt{3}\sin(-\tfrac{3\pi}{4})$$

As for the roots of unity, the sum of all the *n*th roots of a complex number is always zero. Make sure that you understand why this is the case, both algebraically (using the sum of a geometric series) and geometrically (using vector addition).