Section 1: Conditional probability

Notes and Examples

These notes contain sub-sections on:

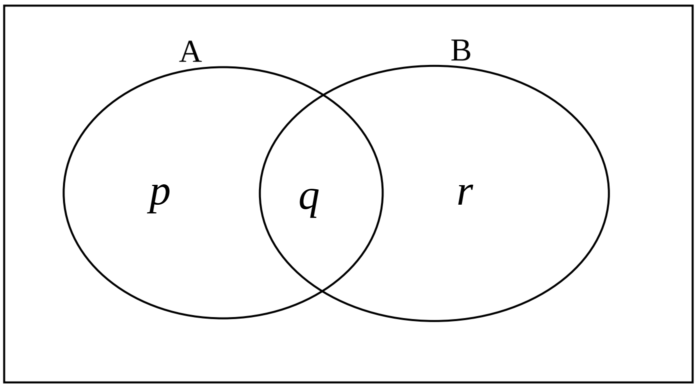
Probability of either one event or another

Sometimes you need to work with probabilities involving two events which are not mutually exclusive.

It is important to realise that in probability work the word **or** has a special meaning.
P(*A* or *B*) means that *A* or *B* or both *A* and B can occur. This is written as P($A \cup B$).

The probability that both A and B occur is written as P($A \cap B$).

The Venn diagram below shows two non-mutually exclusive events A and B.
You can see that $\mathrm{P}(A) = p + q$, and $\mathrm{P}(B) = q + r$.



If you want to find P($A \cup B$), adding $\mathrm{P}(A)$ and $\mathrm{P}(B)$ will give $p + q + q + r$, so q (the probability P($A \cap B$)) has been counted twice.

So $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

Edexcel A level Maths Probability 1 Notes & Examples

There are three different solutions shown in this example. In each case the same things are being done, but different approaches are used. It is useful if you are aware of different strategies, as they may be useful in different situations.

Example 1
In a small sixth form of 50 students Maths and English are the two most popular subjects.
30 students are studying Maths.
25 students are studying English.
10 students are studying Maths and English.
Find the probability that a student chosen at random is studying Maths or English.

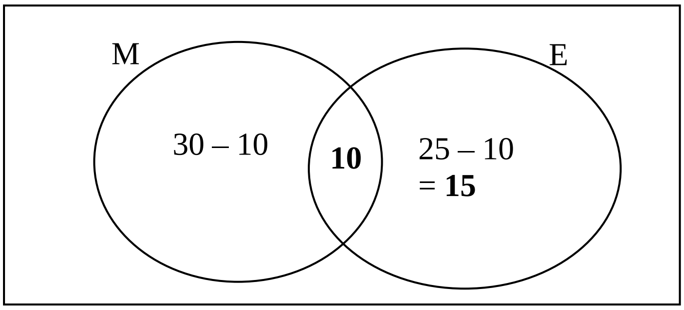
Solution 1: using the formula

Subject	Number of Students	Probability
Maths (M)	30	0.6
English (E)	25	0.5
Both Maths and English	10	0.2

$$\begin{aligned} \mathrm{P}(M \cup E) &= \mathrm{P}(M) + \mathrm{P}(E) - \mathrm{P}(M \cap E) \\ &= 0.6 + 0.5 - 0.2 \\ &= 0.9 \end{aligned}$$

In this example it is obvious you cannot just add 0.6 and 0.5 as you would get an answer bigger than 1.

Solution 2: using a Venn diagram



$$\mathrm{P}(M \cup E) = \frac{20+10+15}{50} = \frac{45}{50} = 0.9$$

Solution 3: using a two-way table

This is the information provided in the question

	English	Not English	Total
Maths	10		30
Not Maths			
Total	25		50

The table can be completed by subtraction

The numbers highlighted are doing Maths or English or both

	English	Not English	Total
Maths	10	20	30
Not Maths	15	5	20
Total	25	25	50

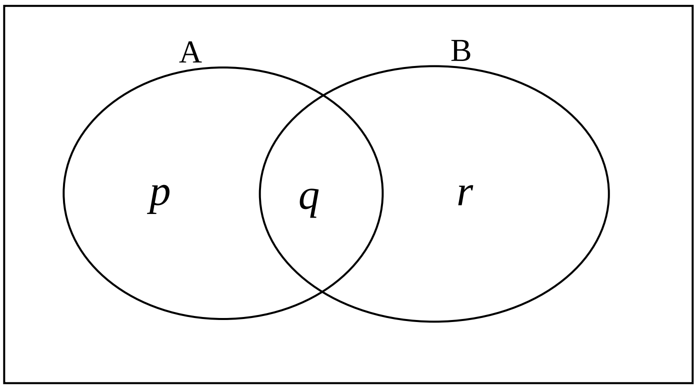
$\mathrm{P}(M \cup E) \ldots \frac{\ldots}{50} = \frac{45}{50} = 0.9$

Alternatively, notice that $\mathrm{P}(M \cup E) = 1 - \mathrm{P}(M' \cap E') = 1 - \frac{5}{50} = \frac{45}{50} = 0.9$

Conditional probability

A conditional probability is the probability that an event occurs given that another event has occurred. The conditional probability of A given B is written as P(A|B).

In the Venn diagram below, the number in B is $q+r$. The number in A that is also in B is q. So the probability of A given B is $\frac{q}{q+r}$.



More generally, the formula for conditional probability is

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Getting information from a table or Venn diagram

The example which follows is similar to the previous example and shows three different ways of dealing with conditional probability. Solution 1 shows a straightforward application of the formula for conditional probability. Solution 2 shows how a table can be used to calculate conditional probabilities quickly. Solution 3 shows how a Venn diagram can be used in a similar way.

Example 2

In a small sixth form of 50 students, Maths and English are the two most popular subjects.
30 students are studying Maths.
25 students are studying English.
10 students are studying Maths and English.
Find the probability that a student studies Maths given that he/she studies English.

Solution 1

Let M be the event that the student studies Maths.
Let E be the event that the student studies English.
We want to find P(M |E).

$$\mathrm{P}(M \mid E) = \frac{\mathrm{P}(M \cap E)}{\mathrm{P}(E)}$$

There are 10 students studying both Maths and English, so $\mathrm{P}(M \cap E) = \frac{10}{50} = \frac{1}{5}$.

There are 25 students studying English, so $\mathrm{P}(E) = \frac{25}{50} = \frac{1}{2}$

Therefore $\mathrm{P}(M \mid E) = \frac{1/5}{1/2} = \frac{1}{5} \times 2 = \frac{2}{5}$

Solution 2

The given information can be presented in a table like the one below.

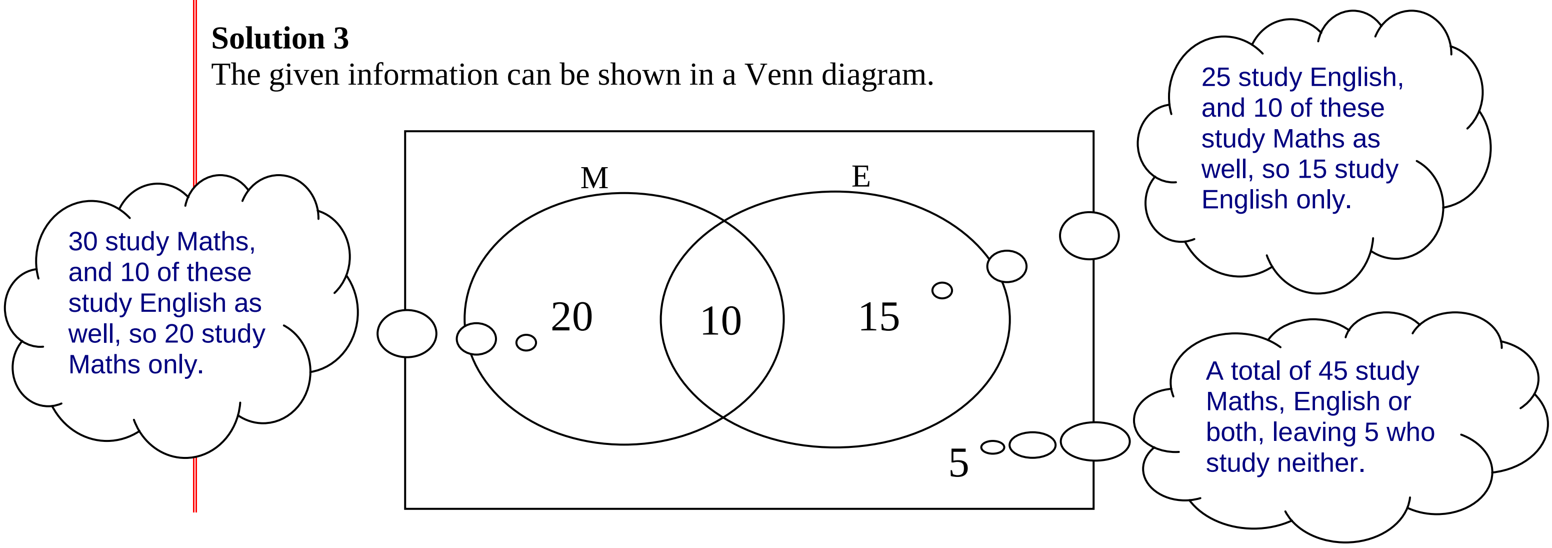
	English	Not English	Total
Maths	10	20	30
Not Maths	15	5	20
Total	25	25	50

To find the probability that a student takes Maths given that he/she takes English, you are considering only the 25 students in the column headed "English".
Out of these 25 students 10 take Maths.

So the probability of a student studying Maths given they study English is $\frac{10}{25} = \frac{2}{5}$.

Solution 3

The given information can be shown in a Venn diagram.



The diagram shows that 25 students study English, and of those 25, 10 study Maths.
So the probability of a student studying Maths given they study English is $\frac{10}{25} = \frac{2}{5}$.

From a table or Venn diagram like the ones in Solution 2 and Solution 3, you can work out many conditional probabilities.

You can find the probability of a student studying English given that they study Maths by looking at the “Maths” row in the table, or considering the Venn diagram.

$$\mathrm{P}(E\,|M) = \frac{10}{30} = \frac{1}{3}$$

You can find the probability of a student studying English given they do not study Maths by looking at the “Not Maths” row in the table, or by considering the Venn diagram (20 do not study Maths, and 15 of these study English).

$$\mathrm{P}(E\,|M') = \frac{15}{20} = \frac{3}{4}$$

Using these last two results, $\mathrm{P}(E\,|M) \neq \mathrm{P}(E\,|M')$
you can see that the event E is not independent of the event M.

Independent and dependent events

If events A and B are **independent**:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B).$$

We can use conditional probability to find out whether two events are independent.

Example 3
The number of students selecting Maths and French is shown in the table.

	French	Not French	Total
Maths	9	21	30
Not Maths	6	14	20
Total	15	35	50

Find the probability that:
(i) a student chosen at random studies Maths
(ii) a student studies Maths given that he/she studies French
(iii) a student studies Maths given that he/she does not study French.
What can you deduce from these results?

Solution
Let M be the event that the student studies Maths.
Let F be the event that the student studies French.

(i) $\quad \mathrm{P}(M) = \frac{30}{50} = \frac{3}{5}$

Look at the "French" column. Out of these 15 students, 9 are studying Maths.

(ii) $\quad \mathrm{P}(M \mid F) = \frac{9}{15} = \frac{3}{5}$

Look at the "Not French" column. Out of these 35 students, 21 are studying Maths.

(iii) $\quad \mathrm{P}(M \mid F') = \frac{21}{35} = \frac{3}{5}$

In this case, $\mathrm{P}(M \mid F) = \mathrm{P}(M \mid F') = \mathrm{P}(M)$
This means that the probability that a student studies Maths is not affected by the choice of whether they study French or not.
Therefore the events M and F are independent.

In general, two events A and B are independent if:

$$\mathrm{P}(A \mid B) = \mathrm{P}(A)$$

Note that in practice you would not need to work out all three probabilities shown in Example 3. Any two of these would be sufficient to show that the events are independent.

Further examples

Example 4
25 interviews were undertaken by a sports centre to research how people kept fit.
Of the 11 men interviewed, 7 preferred team sports to working out in a gym.
8 women preferred working out in a gym to team sports.
Given that a person who prefers working out in a gym is chosen at random, what is the probability that this is a woman?

Solution
Let W be the event that the person selected is a woman.
Let G be the event that the person prefers working out in a gym.

This is an ideal problem for a 2-way table.
Start by writing in the information given in the question.

	Team Sports	Gym	Total
Male	7		11
Female		8	
Total			25

Using subtraction/addition, find the missing values.

	Team Sports	Gym	Total
Male	7	4	11
Female	6	8	14
Total	13	12	25

Looking at the "Gym" column in the table:

$$\mathrm{P}(W\,|\,G) = \frac{8}{12} = \frac{2}{3}$$

You can see that the question becomes quite trivial when you have organised the data.
The table does not have to be restricted to just two categories, although often we will have two categories. Look for situations involving pass/fail, not just for exams, but for example, electrical components.

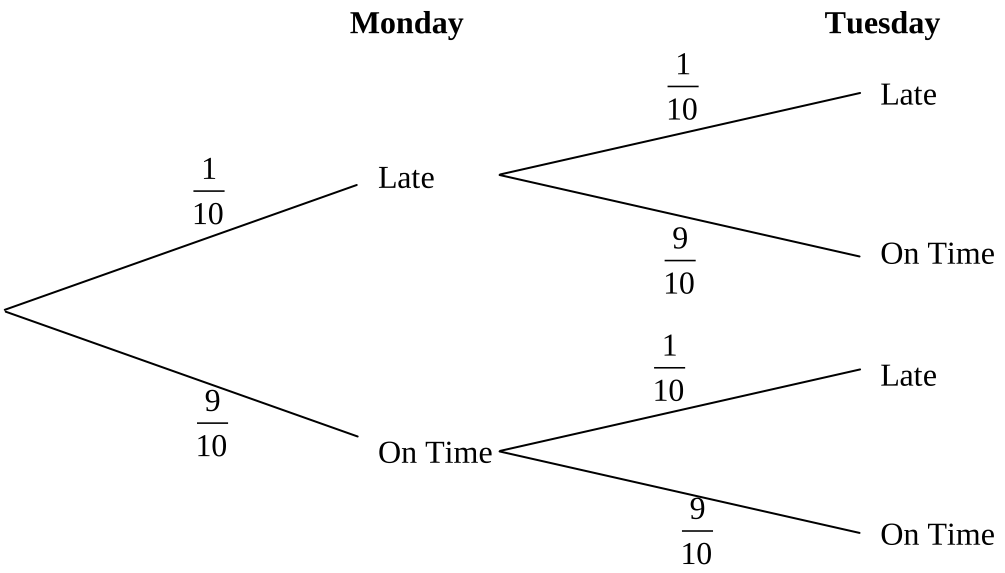
Using a table is not the only way to deal with problems like these. Conditional probability can often follow on from a tree diagram question.

Example 5
A form tutor wants to find the probability that a student, Myles, will not be late on either of Monday or Tuesday, given that he will be on time for at least one of the days. From previous records he finds that:

$$\text{P(Myles is late)} = \frac{1}{10}$$

Assume that Myles' patterns of lateness are independent.

Solution
Let A be the event that Myles was on time for at least one of the days.
Let B be the event that Myles was not late on either day.



$$P(\text{Late, Late}) = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$
$$P(\text{Late, On Time}) = \frac{1}{10} \times \frac{9}{10} = \frac{9}{100}$$
$$P(\text{On Time, Late}) = \frac{9}{10} \times \frac{1}{10} = \frac{9}{100}$$
$$P(\text{On Time, On Time}) = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$$

So the probability Myles will be late on 2 consecutive days is $\frac{1}{100}$.

The probability Myles will be on time on 2 consecutive days is $\frac{81}{100}$.

The probability that Myles will be late on exactly 1 occasion is $\frac{9}{100} + \frac{9}{100} = \frac{18}{100} = \frac{9}{50}$

We require $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$

$P(A)$ = P(on time for at least one of the days)

= 1 – P(late on both days)

$= 1 - \frac{1}{100} = \frac{99}{100}$

Has to satisfy both conditions. In the first Myles is on time on both days, in the second Myles is on time for at least one day. So Myles must be on time on both days.

$P(B \cap A)$ = P(Myles was not late on either day **and** Myles was on time for at least one of the days)

= P(Myles was on time on both days)

$= \frac{81}{100}$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{81}{100} \div \frac{99}{100} = \frac{81}{99}$$

Example 6

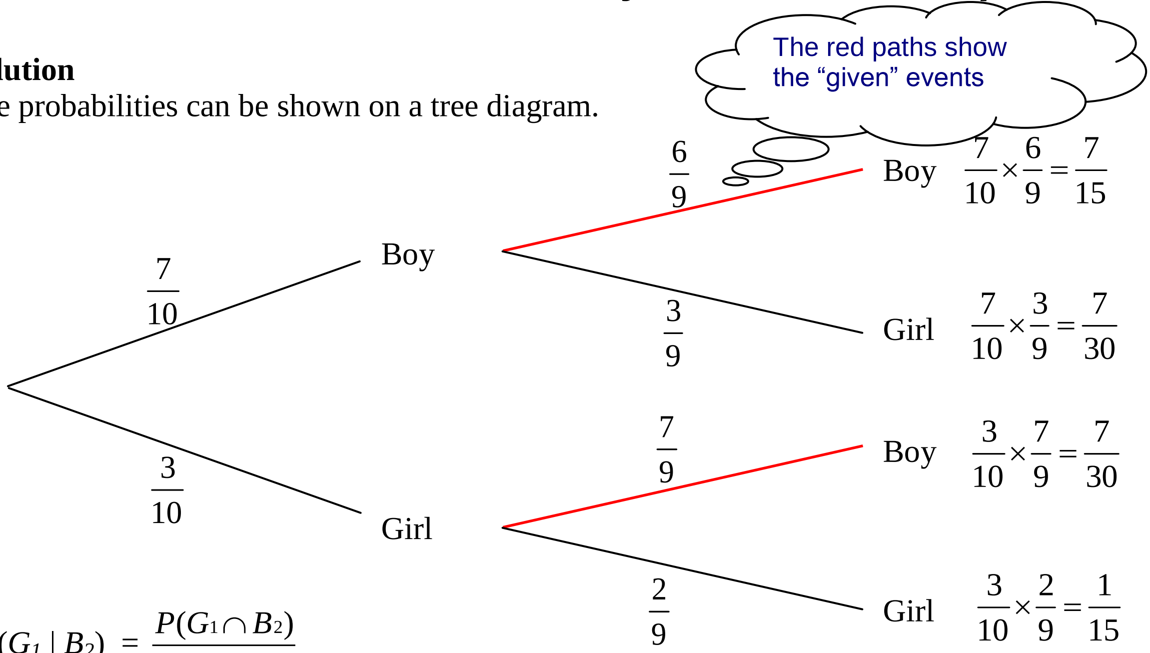
An ‘A’ level teaching group has 7 boys and 3 girls.

2 students are selected at random from this group.

Find the probability that the first student chosen is a girl, given that the second choice is a boy.

Solution

The probabilities can be shown on a tree diagram.



$(\mathrm{P}(G_1 \mid B_2) = \dfrac{P(G_1 \cap B_2)}{P(B_2)}$

$\mathrm{P}(G_1 \cap B_2) = \mathrm{P}(\text{first is a girl and second is a boy}) = \dfrac{7}{30}$

$\mathrm{P}(B_2) = \dfrac{7}{15} + \dfrac{7}{30} = \dfrac{21}{30} = \dfrac{7}{10}$

$\mathrm{P}(G_1 \mid B_2) = \dfrac{7/30}{7/10} = \dfrac{10}{30} = \dfrac{1}{3}$

Notice that Example 6 involves the conditional probability of events out of normal sequence: i.e. the probability of a particular outcome in the first event, given the probability of a particular outcome in the second event. The next example also looks at a problem like this.

Example 7

A Head of Mathematics has analysed the results of students taking AS Maths and A level Maths over the last 10 years at her centre, collecting information on students who have achieved a grade A or B.

She has estimated the following probabilities.

The probability that a student gets an A or B grade on AS Maths is 0.6.

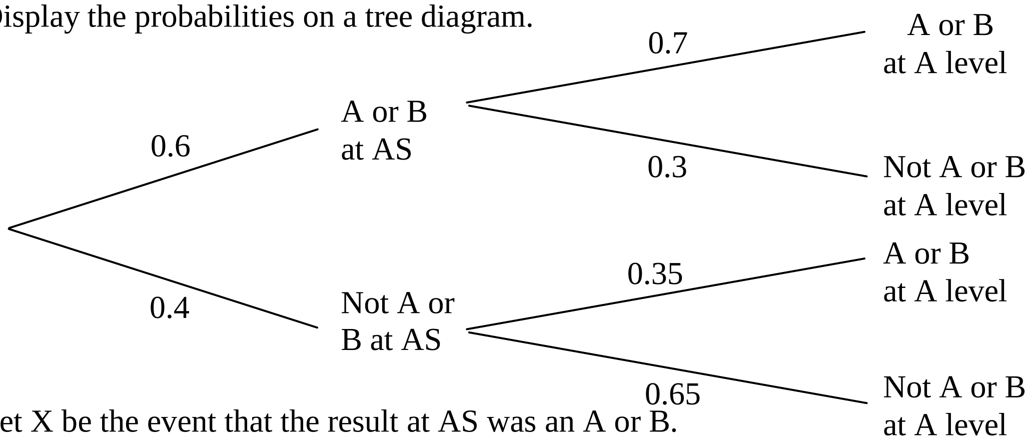
If the student has achieved a grade A or B at AS Maths, the probability that a student gets an A or B grade at A level Maths is 0.7.

If the student has not achieved a grade A or B at AS, the probability that a student gets an A or B grade at A level is 0.35.

Calculate the probability that an A or B grade is achieved at AS, given that an A or B grade has been achieved at A level.

Solution

Display the probabilities on a tree diagram.



Let X be the event that the result at AS was an A or B.
Let Y be the event that the result at A level was an A or B.

We require the probability that an A or B grade is achieved at AS, given that an A or B grade has been achieved at A level. This is P(X | Y).

$$\mathrm{P}(X|\,Y) = \frac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)}$$

$\mathrm{P}(X \cap Y)$ = P(A or B grades in both) = $0.6 \times 0.7 = 0.42$

$\mathrm{P}(Y) = 0.6 \times 0.7 + 0.4 \times 0.35 = 0.42 + 0.14 = 0.56$

$$\mathrm{P}(X \mid Y) = \frac{\mathrm{P}(X \cap Y)}{\mathrm{P}(Y)} = \frac{0.42}{0.56} = \frac{42}{56} = \frac{3}{4}$$