Section 1: Working with probability

Notes and Examples

These notes contain subsections on

- **Notation**
- **Sample space diagrams**
- **Mutually exclusive events: the addition rule**
- **Venn diagrams**
- **Independent events: the multiplication rule**
- **Probability tree diagrams**
- **Expected frequency**
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Notation

As in all mathematics work, it is important that your work is clear and easy to follow. Using correct notation helps!

- P(A) is a quick way of writing down 'the probability of event A occurring.'

- P(H) can be interpreted as a quick way of writing down 'the probability of a head occurring.'

- P(HH) can be interpreted as a quick way of writing down 'the probability of a head followed by a head occurring.'

Try to make your work clear. Not only will your work become easier to mark, but it will also make it easier for you to check your solution and find any errors.

Sample space diagrams

When you are dealing with two events, it can be helpful to use a sample space diagram. This is a useful way of listing all the possible outcomes from two events.

Example 1
A regular tetrahedron has one of the numbers 1, 2, 3, 4 on each face.
This is rolled with an ordinary die.
The score is the sum of the numbers showing.
Find the probability of each score

Solution

DIE score	TETRAHEDRON score			
	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8
5	6	7	8	9
6	7	8	9	10

Careful listing could solve this problem, but with 24 outcomes it is easy to miss out cases or duplicate others. The sample space diagram clearly and efficiently shows all of the possible outcomes.

There are 24 possible outcomes.
You can now work out the probabilities of each score.
For example there are three 8s in the table, so the probability of getting an 8 is $\frac{3}{24}$.

Score	Probability
2	$\frac{1}{24}$
3	$\frac{2}{24}=\frac{1}{12}$
4	$\frac{3}{24}=\frac{1}{8}$
5	$\frac{4}{24}=\frac{1}{6}$
6	$\frac{4}{24}=\frac{1}{6}$
7	$\frac{4}{24}=\frac{1}{6}$
8	$\frac{3}{24}=\frac{1}{8}$
9	$\frac{2}{24}=\frac{1}{12}$
10	$\frac{1}{24}$

It is simple to calculate these probabilities using the sample space diagram.

It is usually best not to simplify the fractions, as if you need to do any further calculations it will be easier to work with a common denominator.

In this case all the separate scores on the die or tetrahedron were equally likely, so it is easy calculate their probabilities. The calculations when the original probabilities are not the same are much more complicated.

Mutually exclusive events: the addition rule

Mutually exclusive events are events which cannot both occur at the same time. Sometimes it is obvious from the context if events are mutually exclusive: for example if A is the event that you score a 6 when you throw a die, and B is the event that you score an odd number when you throw a die, then clearly A and B cannot both occur with one throw of the die, so they are mutually exclusive events.

If A and B are mutually exclusive events, then you can find the probability of either A or B occurring, by adding the probabilities.

$$P(A \cup B) = P(A) + P(B)$$

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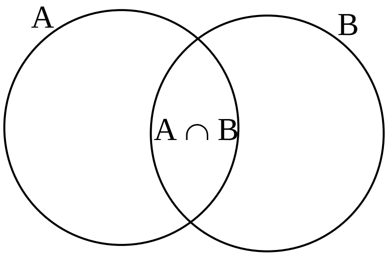
Remember that P(A ∪ B) is the event of A or B occurring, where events A and *B* are mutually exclusive.

Venn diagrams

When you are dealing with events that are not mutually exclusive, you cannot simply add the probabilities. A Venn diagram is often a helpful way to show the information.



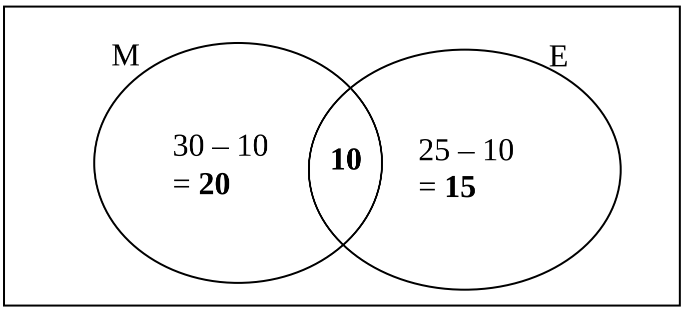
You can see from the diagram above that if you want to find the probability of A or B (i.e. A ∪ B) then if you add P(A) and P(B) then you will count the overlap (P(A ∩ B)) twice. So you need to subtract one of these.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In the example below, two different methods are shown, one using a Venn diagram and the other using a two-way table. In each case the same things are being done, but different approaches are used. It is useful if you are aware of different strategies, as they may be useful in different situations.

Example 2
In a small sixth form of 50 students Maths and English are the two most popular subjects.
30 students are studying Maths.
25 students are studying English.
10 students are studying Maths and English.
Find the probability that a student chosen at random is studying Maths or English.

Solution 1: using a Venn diagram



$$P(M \cup E) = \frac{20+10+15}{50} = \frac{45}{50} = 0.9$$

This is the information provided in the question

Solution 2: using a two-way table

	English	Not English	Total
Maths	10		30
Not Maths			
Total	25		50

The table can be completed by subtraction

The numbers highlighted are doing Maths or English or both

	English	Not English	Total
Maths	10	20	30
Not Maths	15	5	20
Total	25	25	50

$$\mathrm{P}(M \cup E) = \frac{10+20+15}{50} = \frac{45}{50} = 0.9$$

Alternatively, notice that $\mathrm{P}(M \cup E) = 1 - \mathrm{P}(M' \cap E') = 1 - \frac{5}{50} = \frac{45}{50} = 0.9$

Independent events: the multiplication rule

If events *A* and *B* are **independent** events then the outcome of *A* has no influence on the outcome *B*. For example if you get a Head on the first throw of a coin, this does not affect the probability that you get a Head on the second throw of a coin.

If events *A* and *B* are independent:

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$$

Remember that $\mathrm{P}(A \cap B)$ is the event when both *A* and *B* occur.

Example 3

In a game, two dice are thrown.
Let *A* be the event the first die is a 6.
Let *B* be the event the second die is a 6.
Find the probability that you get 2 sixes.

Solution

$$\mathrm{P}(2\text{ sixes}) = \mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

This multiplication rule can be extended to more than two independent events. If there are three independent events *A*, *B* and *C* you multiply the three probabilities

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Probability tree diagrams

When you are dealing with the probability of two or more events, you need ways of displaying all possible events.
A good way to do this is to use a tree diagram.

A tree diagram allows you to highlight all the possible outcomes and systematically work out the corresponding probabilities.

Example 4
A form tutor is investigating the probability that a particular student, Myles, is late on Monday and Tuesday one week.

From previous records he finds that

$$\text{P(Myles is late)} = \frac{1}{10}$$

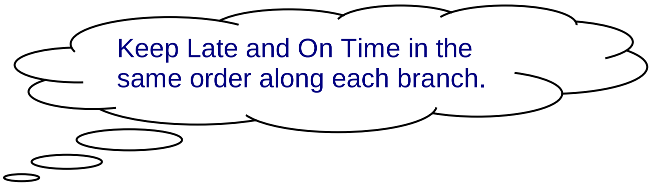
Assuming that Myles' patterns of lateness are independent, find the probability that
(i) Myles is late on both Monday and Tuesday
(ii) Myles is on time on both Monday and Tuesday
(iii) Myles is on time only once in these two days.

Solution

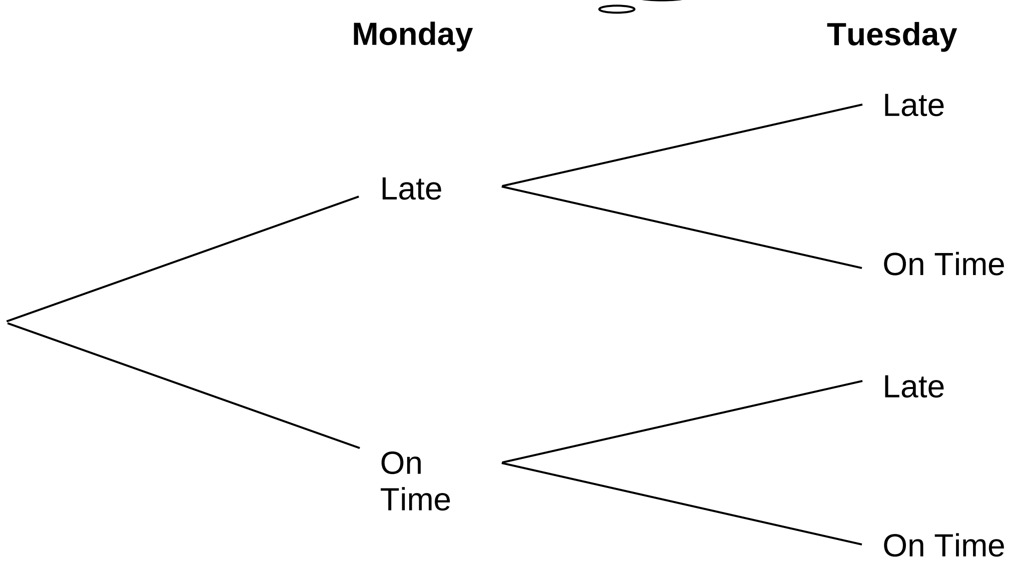
$\text{P(Myles is late)} = \frac{1}{10} \Rightarrow \text{P(Myles is on time)} = \frac{9}{10}$

There are four possible outcomes over two days:

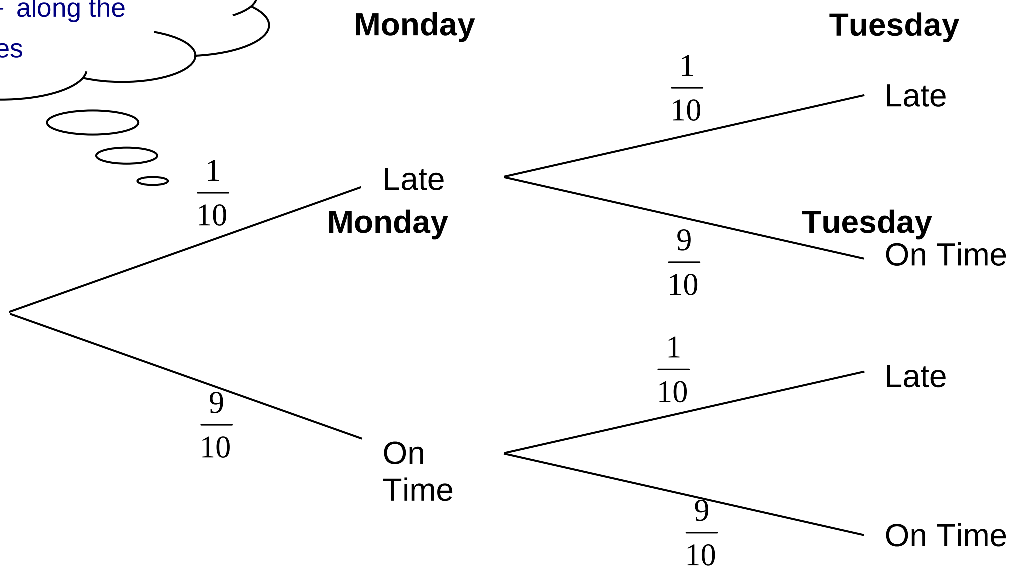
Monday	Tuesday
Late	Late
Late	On time
On time	Late
On time	On time



These are represented clearly by this tree diagram:



Insert $\frac{1}{10}$ and $\frac{9}{10}$ along the relevant branches



Using the multiplication rule you can calculate the probability of the combined events, working across the tree diagram.

$\text{P(Late, Late)} = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ $\qquad$ $\text{P(Late, On Time)} = \frac{1}{10} \times \frac{9}{10} = \frac{9}{100}$

$\text{P(On Time, Late)} = \frac{1}{10} \times \frac{9}{10} = \frac{9}{100}$ $\qquad$ $\text{P(On Time, On Time)} = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$

(i) The probability Myles will be late on both days is $\frac{1}{100}$.

(ii) The probability Myles will be on time on both days is $\frac{81}{100}$.

The events (Late, On Time) and (On Time, Late) both result in Myles being late exactly once.
So to work out P(Myles being late exactly once), add the two probabilities.

(iii) The probability that Myles will be on time only once $= \frac{9}{100} + \frac{9}{100} = \frac{18}{100} = \frac{9}{50}$.

Note that the three possible outcomes: Late twice
On Time twice
Late Once

have probabilities that add up to 1.

$$\frac{1}{100}+\frac{81}{100}+\frac{9}{50}=\frac{100}{100}=1$$

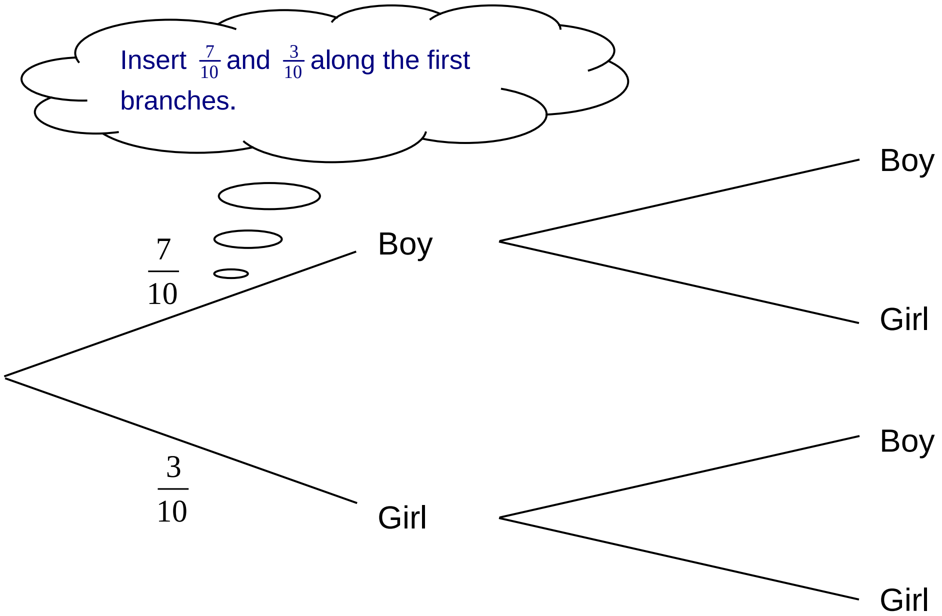
This is because these three outcomes are **exhaustive**. They cover all possible outcomes so it is certain that one of them must occur. The probability of certainty is 1.

In Example 4, the probability of Myles being late on the second day was independent of the probability that he was late on the first day.

Sometimes the probabilities for the second trial are different, especially in selection problems when you select one item and then another. Example 5 shows how this works.

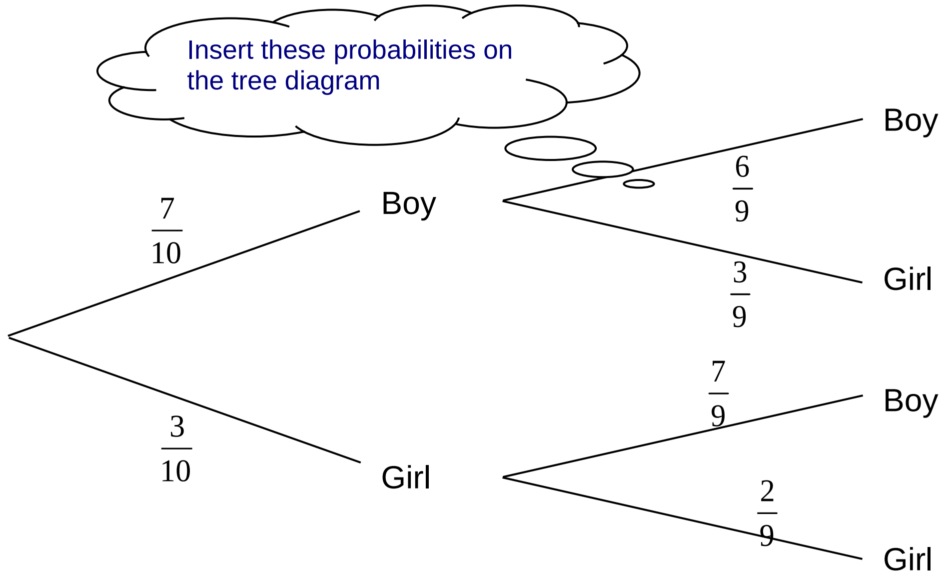
Example 5
An 'A' level teaching group has 7 boys and 3 girls.
2 students are selected at random from this group.
Find the probability that the students selected are
(i) two boys
(ii) two girls
(iii) one boy and one girl.

Solution
When selecting the first student, the probability of selecting a boy is $\frac{7}{10}$ and the probability of selecting a girl is $\frac{3}{10}$.



If a boy is selected first, there are 9 students left: 6 boys and 3 girls.
So the probability that the second student selected is a boy is $\frac{6}{9}$, and the probability that the second student selected is a girl is $\frac{3}{9}$.

If a girl is selected first, there are 9 students left: 7 boys and 2 girls.
So the probability that the second student selected is a boy is $\frac{7}{9}$, and the probability that the second student selected is a girl is $\frac{2}{9}$.



You can now use the multiplication rule to calculate the probabilities of the combined events.

$$\text{P(Boy, Boy)} = \frac{7}{10} \times \frac{6}{9} = \frac{42}{90} = \frac{7}{15}$$

$$\text{P(Boy, Girl)} = \frac{7}{10} \times \frac{3}{9} = \frac{21}{90} = \frac{7}{30}$$

$$\text{P(Girl, Boy)} = \frac{3}{10} \times \frac{7}{9} = \frac{21}{90} = \frac{7}{30}$$

$$\text{P(Girl, Girl)} = \frac{3}{10} \times \frac{2}{9} = \frac{6}{90} = \frac{1}{15}$$

The events (Boy, Girl) and (Girl, Boy) both result in selecting one boy and one girl. So to work out P(1 boy and 1 girl selected), add the two probabilities.

(i) The probability 2 boys will be selected is $\frac{7}{15}$

(ii) The probability 2 girls will be selected is $\frac{1}{15}$

(iii) The probability that one boy and one girl will be selected is $\frac{7}{30} + \frac{7}{30} = \frac{14}{30}$

$$= \frac{7}{15}$$

Note that the three possible outcomes: two boys
two girls
one boy and one girl

have probabilities that add up to 1.

$$\frac{7}{15}+\frac{1}{15}+\frac{7}{15}=1$$

This is because these events are **exhaustive**, as discussed in the previous example.

Notice that in the example above, the two events are not independent. The probability of selecting a boy the second time depends on whether a boy or a girl has been selected the first time. Tree diagrams are particularly helpful for dealing with events which are not independent.

Expected frequency

An estimate of the number of times an event with probability of $\frac{3}{8}$ happens over 400 trials is $\frac{3}{8} \times 400 = 150$.

This is called the expected frequency. This can be a fraction as it represents the average of the times the event will occur.
Do not round to the nearest integer.
e.g. the expected number of heads when a fair coin is tossed 15 times is $\frac{1}{2}\times 15 = 7.5$.

Imagine a simple situation where you pay £1 to play the game.
If you win the game you get your £1 back and a prize of £50.
However if you lose the game you lose the £1 (often referred to as the stake).

Suppose the probability of winning the game is $\frac{1}{100}$. (Note: in reality when playing games like this you rarely know the probability!)

If you win the game you win £50. Since the probability of winning the game is $\frac{1}{100}$, on average you will gain $\frac{1}{100}\times £50 = £0.50$ per game.

If you lose the game you lose £1. Since the probability of losing the game is $\frac{99}{100}$, on average you will lose $\frac{99}{100}\times £1 = £0.99$ per game.

Overall your expected winnings per game is:
£0.50 - £0.99 = -£0.49
The negative value for the winnings means a loss of 49p per game.

This gives you an indication of what will happen over a long period of playing the game.

Clearly, you should decide not to play the game!

You should not play if your aim is simply to make a profit. In reality many games of this type are played for fun, and to raise money for charity.

The complement of an event

The complement of an event A is the event that A does not happen. It is usually denoted as A′.

For example, if event A is ‘getting a 6 when you throw a dice’, then A′ is ‘not getting a 6 when you throw a dice’. If event B is ‘passing a driving test’ then B′ is ‘not passing a driving test’.

Either an event happens or its complement happens, so this means that

$$P(A) + P(A') = 1.$$

You can use this to work out the probability of a complement – since the probability of getting a 6 is $\frac{1}{6}$, the probability of not getting a 6 is $\frac{5}{6}$.

Sometimes it can be easier to find a probability by first finding the probability of the complementary event.

For example, if A is the event of getting at least one six when you throw 5 dice, A' is the event of getting no sixes when you throw 5 dice.
To calculate P(A) it will much be simpler to calculate P(A') first and then use $\mathrm{P}(A)=1-\mathrm{P}(A')$.

Similarly, if B is the event of getting at least one head when you throw 4 coins, B' is the event of getting no heads when you throw 4 coins. To calculate P(B) it will be much simpler to calculate P(B') first and then use $\mathrm{P}(B)=1-\mathrm{P}(B')$.

Example 6

(i) Find the probability of getting at least one six when you throw 5 dice.
(ii) Find the probability of getting at least one head when you throw 4 coins.

Solution

(i) Let A be the event that you get at least one six when you throw 5 dice.
So A' is the event that you get no sixes when you throw 5 dice.
The probability of not getting a 6 is $\frac{5}{6}$.

Using the multiplication rule (since the throws are independent):

$$P(A') = \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6} = \left(\frac{5}{6}\right)^5$$

So $P(A) = 1 - \left(\frac{5}{6}\right)^5 = 0.598$ (3 s.f.)

Round decimals to 3 significant figures and state clearly how you have rounded.

(ii) Let A is the event of getting at least one head when you throw 4 coins,
So A' is the event of getting no heads when you throw 4 coins.
The probability of not getting a Head is $\frac{1}{2}$.
Using the multiplication rule (since the throws are independent):

$$P(A') = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

So $P(A) = 1 - \left(\frac{1}{16}\right) = \frac{15}{16}$