

## Section 1: Introducing the binomial distribution

### Notes and Examples

These notes contain subsections on:

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### The binomial distribution

The binomial distribution is used to model situations where there are a number of different trials, each with two possible outcomes.

To understand the formula for the binomial distribution, suppose you have five trials, each with two possible outcomes, success (S) and failure (F). If you want to find the probability of two successes, you need to think about the ten possible cases

SSFFF	SFSFF
SFFSF	SFFFS
FSSFF	FSFSF
FSFFS	FFSSF
FFSFS	FFFSS

What you are doing is choosing two trials out of the five, to be successes. This is the same process as when you expand  $(x + y)^5$  i.e.  $(x + y)(x + y)(x + y)(x + y)(x + y)$  - to find the coefficient of  $x^2y^3$  you need to know how many ways you can choose an  $x$  from two of the brackets and a  $y$  from the other three. In both these situations, you need to use the binomial coefficient  ${}_5C_2$  to find the number of ways.

So the if the probability of success is  $p$ , the probability of failure is  $1 - p$ , and so the probability of 2 successes from 5 trials is  ${}_5C_2p^2(1 - p)^3$

In general, you need the binomial coefficient  ${}_nC_r$  for  $r$  successes out of  $n$  trials.

So for a random variable  $X \sim B(n, p)$ :

$$P(X = r) = {}_nC_rp^r(1 - p)^{n-r}$$

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## When to use the binomial distribution

It is important that you can identify situations which can be modelled using the binomial distribution.

- There are  $n$  independent trials
- There are just two possible outcomes to each trial, success and failure, with fixed probabilities of  $p$  and  $q$  respectively, where  $q = 1 - p$ .

The discrete random variable  $X$  is the number of successes in the  $n$  trials.  $X$  is modelled by the binomial distribution  $B(n, p)$ . You can write  $X \sim B(n, p)$ .

### Some examples

- $X$  is the number of heads when a coin is tossed 20 times.  
Each coin toss represents a trial, so  $n = 20$ .  
The probability of success (i.e. getting a head) is  $\frac{1}{2}$  so  $p = \frac{1}{2}$   
 $X \sim B(20, \frac{1}{2})$
- $X$  is the number of sixes when a die is thrown 10 times.  
Each throw of the die represents a trial, so  $n = 10$   
The probability of success (i.e. getting a six) is  $\frac{1}{6}$  so  $p = \frac{1}{6}$   
 $X \sim B(10, \frac{1}{6})$

You might think that there are six possible outcomes to throwing a die. However, since you are only interested in whether or not you get a six, there are two outcomes, getting a six and not getting a six.

- A particular test has a pass rate of 60%.  $X$  is the number of students who fail the test out of a class of 30.  
Each student represents a trial, so  $n = 30$ .  
The probability of success (i.e. failing the test) is 40% so  $p = 0.4$   
 $X \sim B(30, 0.4)$

It seems strange to talk about failing a test as “success”! However, since  $X$  is the number of failures, then a “successful” trial is a failure!

Be careful to identify the “trials” correctly. For example, suppose that you were looking at the number of girls in families with three children. If  $X$  is the number of girls in a family with three children, then there are 3 trials and the probability of “success” (i.e. having a girl) is  $\frac{1}{2}$ , and so  $X \sim B(3, \frac{1}{2})$ . However, if you look at 20 families each with three children, and  $X$  is the number of families with three girls, then there are 20 trials and the probability of success (i.e. having three girls) is  $\frac{1}{8}$ , and so  $X \sim B(20, \frac{1}{8})$ .

The Explore resource [Binomial distributions](#) shows simulations of binomial distributions, together with graphical representations of the binomial distribution.



## Finding binomial probabilities

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Your calculator should be able to work out binomial probabilities for you. It should give you options for finding  $P(X = r)$  or  $P(X \leq r)$  - you need to enter values of  $n$ ,  $p$  and  $r$ .

Sometimes it is just as quick to work out  $P(X = r)$  by using the formula

$P(X = r) = {}_n C_r p^r (1 - p)^{n-r}$ , rather than going through a calculator menu system. Make sure you can do this either way. However, finding a cumulative probability of the form  $P(X \leq r)$  is usually best done using the calculator function.

Sometimes you will need to find a probability of the form  $P(X \geq r)$ . Since your calculator will only give  $P(X \leq r)$ , you need to use subtraction to work it out – for example  $P(X \geq 4) = 1 - P(X \leq 3)$ .

In Example 1, the data is presented using the  $B(n, p)$  notation.



## Example 1

$X \sim B(10, 0.4)$ .

Find the following probabilities:

- (i)  $P(X = 1)$
- (ii)  $P(X = 0)$
- (iii)  $P(X \geq 2)$

## Solution

$X \sim B(10, 0.4)$ . So:  $n = 10$ ,  $p = 0.4$ ,  $q = 0.6$

(i) 
$$\begin{aligned} P(X = 1) &= {}_{10}C_1 \times 0.4^1 \times 0.6^9 \\ &= 10 \times 0.4^1 \times 0.6^9 \\ &= 0.0403 \text{ (3 s.f.)} \end{aligned}$$

(ii) 
$$\begin{aligned} P(X = 0) &= {}_{10}C_0 \times 0.4^0 \times 0.6^{10} \\ &= 1 \times 1 \times 0.6^{10} \\ &= 0.00605 \text{ (3 s.f.)} \end{aligned}$$

(iii) 
$$\begin{aligned} P(X \geq 2) &= 1 - P(X \leq 1) \\ &= 1 - 0.046 \\ &= 0.954 \text{ (3 s.f.)} \end{aligned}$$

Your calculator should be able to find these probabilities, but you may feel it is just as quick to write it out.

You can find cumulative probabilities like  $P(X \leq 1)$  from your calculator.

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## Example 2

Using recent data provided by the low-cost airline Brianair, the probability of a flight arriving on time is estimated to be 0.9.

On four different occasions I am taking a flight with Brianair.

- What is the probability that I arrive on time on all four flights?
- What is the probability that I arrive on time on exactly two occasions?
- What is the probability that I arrive on time on at least one occasion?

## Solution

Let  $X$  be the number of times a flight is on time.

$n = 4, p = 0.9, q = 0.1$  so  $X \sim B(4, 0.9)$ .

- $P(\text{all four flights arrive on time}) = P(X = 4) = {}_4C_4 \times 0.9^4 \times 0.1^0$   
 $= 1 \times 0.9^4 \times 1$   
 $= 0.656$  (3 s.f.)
- $P(\text{exactly two flights arrive on time}) = P(X = 2) = {}_4C_2 \times 0.9^2 \times 0.1^2$   
 $= 6 \times 0.9^2 \times 0.1^2$   
 $= 0.0486$  (3 s.f.)
- $P(\text{at least one flight arrives on time}) = P(X \geq 1) = 1 - P(X = 0)$   
 $P(X = 0) = {}_4C_0 \times 0.9^0 \times 0.1^4 = 1 \times 1 \times 0.1^4 = 0.0001$   
 $P(X \geq 1) = 1 - P(X = 0)$   
 $= 1 - 0.0001$   
 $= 0.9999$



## Example 3

Using recent data provided by the low-cost airline Lyingair, the probability of a flight arriving on time is estimated to be 0.7.

Every week I take four flights with Lyingair.

- Find the expectation of the number of times that I will be on time.
- In a particular week, what is the probability that I arrive at my destination on time for all 4 flights?
- What is the probability that in 3 weeks of travelling, exactly one of the weeks has at least one late arrival?
- What is the probability that in 3 weeks of travelling, at least one of the weeks has at least one late arrival?

## Solution

Let  $X$  be the number of times in a particular week that a flight is on time.

$n = 4, p = 0.7, q = 0.3$

$X \sim (4, 0.7)$ .

- What is the expected value?  
Expected value  $= np = 4 \times 0.7 = 2.8$

Do not round this value. Although the flight cannot be on time on 2.8 occasions, this gives us an indication of what will happen over a long period of time, with repeated samples.



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(ii) 
$$P(X = 4) = {}_4C_4 \times 0.7^4 \times 0.3^0$$
$$= 1 \times 0.7^4 \times 1$$
$$= 0.2401$$

- (iii) The probability that in one week all the flights are on time is 0.2401 from (ii).  
Let  $Y$  be the number of weeks that all flights on time.  
 $Y$  is binomial with  $n = 3$  and  $p = 0.2401$   
 $Y \sim B(3, 0.2401)$ .

Note: the flights can be all on time during a week, or there is at least one delay, giving a Binomial situation.

Probability that in 3 weeks of travelling one of the weeks has at least one late arrival is  $P(Y = 2)$ .

$$P(Y = 2) = {}_3C_2 \times 0.2401^2 \times 0.7599^1$$
$$= 0.131 \text{ (3 s.f.)}$$

$P(Y = 2)$  is the probability that flights are on time for two of the three weeks, which is the same as having at least 1 late arrival in 1 week.

- (iv) First find the probability that there are no late arrivals for any of the weeks, i.e. the probability that  $Y = 3$ .  
$$P(Y = 3) = {}_3C_3 \times 0.2401^3 \times 0.7599^0$$
$$= 0.0138 \text{ (3 s.f.)}$$

The probability that in 3 weeks of travelling at least one of the weeks has at least one late arrival =  $1 - 0.0138 = 0.986$  (3 s.f.)

Very careful reading of the question is needed at this stage!!

In parts (iii) and (iv) you are using a previous answer as the probability in a new binomial situation.

## Finding an unknown sample size



### Example 4

Of the students in a school, 25% travel to school by bus. Students are selected at random. How many students must be selected so that the probability that there is at least one student travelling to school by bus is greater than 0.95?



### Solution

Let  $X$  be the number of students travelling by bus in the sample.

$$n = ?, p = \frac{1}{4}, q = \frac{3}{4}$$

This means "the probability that at least one student travels by bus is greater than 0.95" – the statement in the question.

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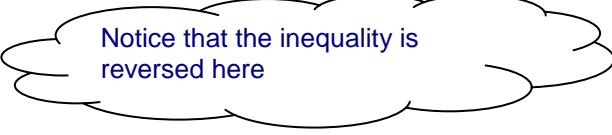
$$X \sim B\left(n, \frac{1}{4}\right).$$

$$P(X \geq 1) > 0.95$$

$$1 - P(X = 0) > 0.95$$

$$1 - 0.95 > P(X = 0)$$

$$P(X = 0) < 0.05$$



Notice that the inequality is reversed here

$$P(X = 0) = \left(\frac{3}{4}\right)^n$$

$$\Rightarrow \left(\frac{3}{4}\right)^n < 0.05$$

Using trial and improvement:

$$\left(\frac{3}{4}\right)^{10} = 0.0563 \quad \text{which is too big}$$

$$\left(\frac{3}{4}\right)^{11} = 0.0422 \quad \text{which is smaller than 0.05.}$$

So the required minimum number of students to be selected is 11.