Section 4: Finding distances

Notes and Examples

These notes contain subsections on

Finding the distance of a point from a plane

The shortest distance of a point A to a plane is the distance AP where AP is a line perpendicular to the plane and P is a point on the plane.

The distance of a point (α, β, γ) from a plane $n_1x + n_2y + n_3z + d = 0$ is given by the formula

$$\left|\frac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{n_1^2 + n_2^2 + n_3^2}}\right|$$

You can prove this formula by thinking about the line $\mathbf{r} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \lambda \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, which goes through the point (α, β, γ) and is perpendicular to the plane. You can find the intersection of the line and the plane, and then find the distance from the point (α, β, γ) to the intersection point.

Example 1 shows this formula being applied.

Example 1
Find the distance of the point (2, -2, 4) from the plane $2x + y - 3z = 4$.

Solution
The equation of the plane can be written as $2x + y - 3z - 4 = 0$, so $a = 2$, $b = 1$, $c = -3$ and $d = -4$.

Distance of point from plane $= \dfrac{|(2\times 2) + (1\times -2) + (-3\times 4) - 4|}{\sqrt{2^2 + 1^2 + 3^2}} = \dfrac{14}{\sqrt{14}} = \sqrt{14}$

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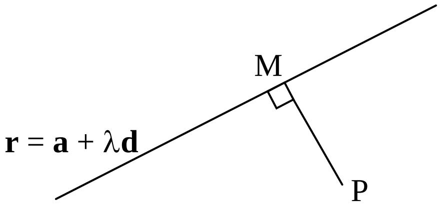
The distance of a point from a line

The distance of a point from a line is the perpendicular distance from the point to the line, which is the shortest possible distance.



To find the distance of a point P from a line with equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$, you need to find a point M which is on this line, and for which the vector $\overrightarrow{\mathbf{PM}}$ is perpendicular to **d**.

Example 2 shows how this can be done.

Example 2

Find the distance of the point P (2, 1, -2) from the line $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-2}{3}$.

Solution

The equation of the line can be written as $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

Let M be the point on the line which is closest to P.

The position vector of M is of the form $\overrightarrow{\mathbf{OM}} = \begin{pmatrix} 1+2\lambda \\ -1-\lambda \\ 2+3\lambda \end{pmatrix}$.

So $\overrightarrow{\mathbf{PM}} = \overrightarrow{\mathbf{OM}} - \overrightarrow{\mathbf{OP}} = \begin{pmatrix} 1+2\lambda \\ -1-\lambda \\ 2+3\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1+2\lambda \\ -2-\lambda \\ 4+3\lambda \end{pmatrix}$

The vector $\overrightarrow{\mathbf{PM}}$ is perpendicular to the line with direction vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

so $\begin{pmatrix} -1+2\lambda \\ -2-\lambda \\ 4+3\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 0$

$$2(-1+2\lambda)-(-2-\lambda)+3(4+3\lambda)=0$$
$$-2+4\lambda+2+\lambda+12+9\lambda=0$$
$$14\lambda=-12$$
$$\lambda=-\frac{6}{7}$$

So $\overrightarrow{\mathbf{PM}} = \begin{pmatrix} -1-\frac{12}{7} \\ -2+\frac{6}{7} \\ 4-\frac{18}{7} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -19 \\ -8 \\ 10 \end{pmatrix}$

$\left|\overrightarrow{\mathbf{PM}}\right| = \frac{1}{7}\sqrt{19^2+8^2+10^2} = \frac{1}{7}\sqrt{525} = \frac{5}{7}\sqrt{21}$

In the example above it is quickest to substitute for λ into the expression for $\overrightarrow{\mathbf{PM}}$ already found, to find the distance PM. You might also need to find the coordinates of M itself, in which case you would need to substitute for λ into the expression for $\overrightarrow{\mathbf{OM}}$. In this example, the coordinates of M turn out to be $\left(-\frac{5}{7}, -\frac{1}{7}, -\frac{4}{7}\right)$.

The distance between two parallel lines

Two lines in 3D space which do not meet are either parallel or skew. You will see how to find the distance between skew lines in Example 4. However, finding the distance between parallel lines is equivalent to finding the shortest distance between any point on one line and the other line, so you can use the same method as in Example 2.

Example 3

Show that the lines $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\frac{x+1}{-2} = y-3 = \frac{z+2}{-1}$ are parallel and find the distance between them.

Solution

The second line can be written as $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$.

Since $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, the direction vectors are parallel and so the lines are parallel.

Let P be the point (1, 2, 0) which is on the first line.
Let M be the point on the second line which is closest to P.

So $\overrightarrow{\mathbf{OM}} = \begin{pmatrix} -1-2\mu \\ 3+\mu \\ -2-\mu \end{pmatrix}$ and $\overrightarrow{\mathbf{PM}} = \begin{pmatrix} -1-2\mu \\ 3+\mu \\ -2-\mu \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2-2\mu \\ 1+\mu \\ -2-\mu \end{pmatrix}$

$$\overrightarrow{\mathbf{PM}}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2-2\mu \\ 1+\mu \\ -2-\mu \end{pmatrix}.\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$-4-4\mu-1-\mu-2-\mu = 0$$

$$6\mu = -7$$

$$\mu = -\tfrac{7}{6}$$

$$\overrightarrow{\mathbf{PM}} = \begin{pmatrix} -2+\frac{14}{6} \\ 1-\frac{7}{6} \\ -2+\frac{7}{6} \end{pmatrix} = \begin{pmatrix} \frac{2}{6} \\ -\frac{1}{6} \\ -\frac{5}{6} \end{pmatrix}$$

So $\left|\overrightarrow{\mathbf{PM}}\right| = \frac{1}{6}\sqrt{2^2+1^2+5^2} = \frac{1}{6}\sqrt{30}$

The distance between two skew lines

To find the distance between two skew lines, you need to find point P on one line and point Q on the other such that PQ is perpendicular to both lines. Example 4 shows how this is done.

Example 4
Find the shortest distance between the two skew lines

$$\frac{x-1}{2} = \frac{y+1}{1} = \frac{z+2}{1} \text{ and } \frac{x-3}{1} = \frac{y+2}{-2} = \frac{z-7}{3}$$

Solution

The lines can be written as $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Let P be a point on the first line, so $\overrightarrow{\mathbf{OP}} = \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ -2+\lambda \end{pmatrix}$.

Let Q be a point on the second line, so $\overrightarrow{\mathbf{OQ}} = \begin{pmatrix} 3+\mu \\ -2-2\mu \\ 7+3\mu \end{pmatrix}$

Then $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{OQ}} - \overrightarrow{\mathbf{OP}} = \begin{pmatrix} 3+\mu \\ -2-2\mu \\ 7+3\mu \end{pmatrix} - \begin{pmatrix} 1+2\lambda \\ -1+\lambda \\ -2+\lambda \end{pmatrix} = \begin{pmatrix} 2+\mu-2\lambda \\ -1-2\mu-\lambda \\ 9+3\mu-\lambda \end{pmatrix}$

$\overrightarrow{\mathbf{PQ}}$ is perpendicular to the first line, so

$$\begin{pmatrix} 2+\mu-2\lambda \\ -1-2\mu-\lambda \\ 9+3\mu-\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$4+2\mu-4\lambda-1-2\mu-\lambda+9+3\mu-\lambda = 0$$

$$-6\lambda+12+3\mu = 0$$

$$2\lambda-\mu = 4$$

$\overrightarrow{\mathbf{PQ}}$ is perpendicular to the second line, so

$$\begin{pmatrix} 2+\mu-2\lambda \\ -1-2\mu-\lambda \\ 9+3\mu-\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 0$$

$$2+\mu-2\lambda+2+4\mu+2\lambda+27+9\mu-3\lambda = 0$$

$$-3\lambda+31+14\mu = 0$$

$$3\lambda-14\mu = 31$$

Solving these equations simultaneously gives $\lambda = 1, \mu = -2$.

So $\overrightarrow{\mathbf{PQ}} = \begin{pmatrix} 2-2-2 \\ -1+4-1 \\ 9-6-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$

$\left|\overrightarrow{\mathbf{PQ}}\right| = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$

So the distance between the two lines is $2\sqrt{3}$