Edexcel AS Further Mathematics Vectors

integral

Section 2: The vector equation of a line

Notes and Examples

These notes contain subsections on

- The vector equation of a line in two dimensions
- Vector equation of a line in three dimensions
- Cartesian equation of a line in three dimensions
- Special cases of a cartesian equation of a line in three dimensions
- The angle between two lines
- Finding the intersection of two lines in two dimensions
- The intersection of two lines in three dimensions

The vector equation of a line in two dimensions

To write down the cartesian equation of a line you need to know:

- the coordinates of one point on the line
- the gradient of the line (or the coordinates of 2nd point)

To write the vector equation of a line you need to know:

- the position vector of one point on the line
- the direction of the line (or the position vector of a 2nd point)

The **general vector equation** of a line AB is:

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$$

This is the position vector of a point on the line. This is like the '+c' part of a cartesian equation.

This gives the direction vector of the line. This is like the 'm' part of a cartesian equation.

Each different point on the line corresponds to a different value of the parameter λ.

Note: the vector equation of a line (like its cartesian equivalent) can be written in many different forms.

If A and B are the points with position **a** and **b,** you can write the vector equation of a line as:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

which is the same as:

$$\mathbf{r} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$$

Example 1
Find the vector equation of the line joining the points A (5, -2) and B (1, 4).

Solution

$\mathbf{a} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

So $\mathbf{r} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \lambda\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix}\right)$

$\Rightarrow \mathbf{r} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -4 \\ 6 \end{pmatrix}$

You can also write this as:

$$\mathbf{r} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Moving in the direction '4 left and 6 up' is the same as moving in the direction '2 left and 3 up'

or even

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

If you start at the point (5, -2) and move '2 left and 3 up' you get to (3, 1)

Example 2
Find the vector equation of the line through the points A (-4, -1) parallel to 4**i** + 2**j**.

Solution

$\mathbf{a} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ and the line is in the direction $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$

So $\mathbf{r} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$\Rightarrow \mathbf{r} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow \mathbf{r} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The vector equation of a line in three dimensions

The vector equation of a line in 3-D is the same as in 2-D:

$$\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$$

Example 3
The points A and B have coordinates (-1, 2, -3) and (0, -2, 1) respectively.
Find the equation of the line AB.

Solution

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix}$$

The vector equation of a line is $\mathbf{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

So: $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix}$

Cartesian equation of a line in three dimensions

The vector equation of a line:

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

can be expressed in cartesian form as:

$$\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$$

Note there are two '=' signs!

Example 4

Write down the cartesian equation of the line $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$

Solution

Substituting into the general form:

$$\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$$

we have:

$$\frac{x-(-2)}{3} = \frac{y-1}{-5} = \frac{z-3}{1}$$

this can be written as:

$$\frac{x+2}{3} = \frac{1-y}{5} = z-3$$

Example 5

Write down the vector equation of the line $\dfrac{3-x}{2} = y-4 = \dfrac{z}{5}$

Solution

Write the equation in the same form as:

$$\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$$

we have:

$$\frac{3-x}{2} = y-4 = \frac{z}{5} \Rightarrow \frac{x-3}{-2} = \frac{y-4}{1} = \frac{z-0}{5}$$

So the vector equation is:

$$\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

Special cases of a cartesian equation of a line in 3-D

When the direction vector of a line in 3-D contains one zero the line is running parallel to either the *x-y* plane, *x-z* plane or the *y-z* plane.
In this case, the Cartesian equation is written slightly differently as is shown in the next example.

Example 6

Write down the cartesian equation of the line $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$

In the line the *x* coordinate is always 4 as the line isn't 'moving' in the *x* direction…

Solution

Substituting into the general form:

$$\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$$

we have:

$$\frac{x-4}{0} = \frac{y-(-1)}{-2} = \frac{z-2}{3}$$

…this means that you would be dividing by 0 – which is undefined…

this can be written as:

$$x = 4 \text{ and } \frac{-(y+1)}{2} = \frac{z-2}{3}$$

…so we write the equation of the line like this.

The next example shows you how to treat two zeros in the direction vector.

Example 7

Write down the cartesian equation of the line $\mathbf{r} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$

In the line the x coordinate is always -3 and the z coordinate is always 1 as the line isn't 'moving' in the x or z directions…

Solution

Substituting into the general form:

$$\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$$

we have:

$$\frac{x-(-3)}{0} = \frac{y-2}{4} = \frac{z-1}{0}$$

…this means that the y co-ordinate could be any value …

this can be written as:

$$x = -3 \text{ and } z = 1$$

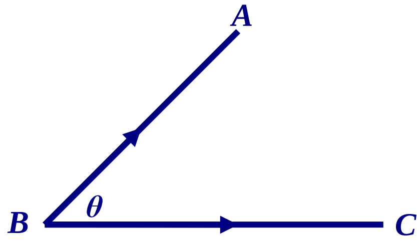
…so we write the equation of the line like this.

The angle between two lines

To find the angle between two lines simply find the angle between their two direction vectors.

Note:

- Two lines in 3-D may not touch but you can still work out the angle between them in the usual way.
- To find $\angle ABC$ you need to work out the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.



Example 8

Find the angle between the lines $\mathbf{r} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

Solution

Find the angle between the two direction vectors $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ using the formula:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\begin{pmatrix}3\\1\\-2\end{pmatrix}.\begin{pmatrix}1\\0\\-3\end{pmatrix}=3\times1+1\times0+(-2)\times(-3)=3+0+6=9$$

The length of $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ is: $\sqrt{3^2+1^2+(-2)^2}=\sqrt{9+1+4}=\sqrt{14}$

The length of $\begin{pmatrix}1\\0\\-3\end{pmatrix}$ is: $\sqrt{1^2+0^2+(-3)^2}=\sqrt{1+0+9}=\sqrt{10}$

So $\cos\theta=\dfrac{9}{\sqrt{14}\sqrt{10}}=0.7606...$

$\Rightarrow \theta=40.5°$ to 3.s.f.

Finding the intersection of two lines in two dimensions

You need to be able to convert between the vector and cartesian equations of a line.

Step 1: Equate the two equations,
Step 2: Write down two equations in λ and μ,
Step 3: Solve these equations simultaneously,
Step 4: Substitute back into the original equations.

The following example will show you how to do this.

Example 9
Find the position vector of the point where the lines:

$$\mathbf{r}=\begin{pmatrix}-3\\-2\end{pmatrix}+\lambda\begin{pmatrix}4\\1\end{pmatrix} \text{ and } \mathbf{r}=\begin{pmatrix}-2\\1\end{pmatrix}+\mu\begin{pmatrix}1\\3\end{pmatrix} \text{ intersect.}$$

Solution
Step 1:

$$\begin{pmatrix}-3\\-2\end{pmatrix}+\lambda\begin{pmatrix}4\\1\end{pmatrix}=\begin{pmatrix}-2\\1\end{pmatrix}+\mu\begin{pmatrix}1\\3\end{pmatrix}$$

The two lines meet at the point where they are equal to each other.

Step 2: Reading across:

x: $\quad -3+4\lambda=-2+\mu \quad$ ①

y: $\quad -2+\lambda=1+3\mu \quad$ ②

You now have two equations and two unknowns. Solve them simultaneously to find either λ or μ.

Step 3:

4×② $\quad -8+4\lambda=4+12\mu \quad$ ③

①-③

$$\begin{array}{rl} & -3+4\lambda = -2+\mu \\ - & -8+4\lambda = 4+12\mu \\ \hline & 5+0 = -6-11\mu \end{array}$$

So:

$$5 = -6-11\mu$$
$$\Rightarrow -11\mu = 11$$
$$\Rightarrow \mu = -1$$

Step 4: $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

So the lines intersect at the point with position vector $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

The intersection of two lines in three dimensions

In two dimensions, two lines which are not parallel will always intersect. This is not the case in three dimensions.

Two lines in three dimensions which do not intersect and are not parallel are called skew lines.

Example 10

Lines l_1 and l_2 have vector equations $\mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ respectively. Do these lines intersect?

Solution

The lines are clearly not parallel as their directions are not parallel (if they were then their direction vectors $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ would be multiples of one-another).

As they are not parallel, we must establish whether or not they are skew.

If the lines meet then their parametric equations can be solved simultaneously i.e. there will exist values of λ and μ which will satisfy each of these equations simultaneously:

$$4+5\lambda = -5+2\mu \quad \text{(A)}$$
$$3+2\lambda = 4-\mu \quad \text{(B)}$$
$$3\lambda = 1 \quad \text{(C)}$$

Note that different symbols must be used to denote the parameter in the two lines. This must be done in order to avoid confusion. If the same symbol were used it would imply that the parameters in each line always have equal values, which is certainly not true.

(C) $\Rightarrow \lambda = \frac{1}{3}$; substitute in (B) $\Rightarrow \mu = \frac{1}{3}$.

Substituting $\lambda = \frac{1}{3}$ and $\mu = \frac{1}{3}$ into (A) $\Rightarrow \frac{17}{3} = \frac{-13}{3}$.

This is clearly a contradiction so the equations cannot be solved simultaneously and the lines must be skew.

In the example above, if the lines did meet, the point of intersection could be found by substituting one or other of the parameters into the appropriate parametric equation.