Section 1: The scalar product

Notes and Examples

These notes contain subsections on

- The scalar product
- Finding the angle between two vectors

The scalar product

The **scalar product** (or **dot product**) of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ is written as $\mathbf{a}\cdot\mathbf{b}$ and is found by:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$$

In the same way, when working with vectors in three dimensions, the scalar product of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is found by:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Example 1

Work out (i) $\begin{pmatrix} 5 \\ -3 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

(ii) $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}.\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

Solution

(i) $\begin{pmatrix} 5 \\ -3 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ -2 \end{pmatrix} = 5\times(-1) + (-3)\times(-2) = -5 + 6 = 1$

(ii) $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}.\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = (2\times 4) + (0\times -2) + (-3\times 1) = 8 + 0 - 3 = 5$

Finding the angle between two vectors

The angle between two vectors is found using the formula:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

This is also written as: $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

The next example shows you how to use this formula to find the angle between two vectors.

Example 2

Work out the angle between the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j}$

Solution

Use the formula $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

$\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 2\times1 + (-3)\times4 = 2 - 12 = -10$

$|\mathbf{a}| = \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

$|\mathbf{b}| = \sqrt{1^2 + 4^2} = \sqrt{1+16} = \sqrt{17}$

Substituting into $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$:

$$\cos\theta = \frac{-10}{\sqrt{13}\sqrt{17}} = -0.6726...$$

So $\theta = 132.3°$ to 1 d.p.

The angle between two 3-D vectors can be found in the same way.

When two lines are perpendicular to each other then:

$$\theta = 90° \Rightarrow \cos\theta = 0 \Rightarrow \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = 0$$

If $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = 0$ then $\mathbf{a}\cdot\mathbf{b} = 0$

This is a very important result and means that to show that two vectors are perpendicular you only need to show that the scalar product is 0.

Example 3

Show that the vectors $\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ are perpendicular.

Solution

If two vectors are perpendicular then $\mathbf{a}\cdot\mathbf{b} = 0$

$\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = (2\times3) + (-4\times2) + (1\times2) = 6 - 8 + 2 = 0$ as required.