Section 2: Proof by induction

Notes and Examples

These notes contain subsections on
- Types of proof: a reminder
- Proof by induction
- Other types of proof by induction

Types of proof: a reminder

You have probably already met the idea of proof in your study of mathematics.

As a reminder, here are some of the basic ideas about proof:
- To prove something in mathematics, you need to show that it is true for all possible cases. For example, if you wanted to prove that the sum of the first n odd numbers is given by n^2, you could check as many different values of n as you like, or you could get a computer to check all values of n up to a very large value, but this would still not prove the result. You might feel pretty confident that the result was correct, but you would not have a proof.
- To disprove something in mathematics, you only need to find one example for which it is not true. This is called a *counterexample*.

There are several different types of proof which you may have already come across:
- Proof by exhaustion: this is when you check all possible cases. You can't do this if there is an infinite number of cases, as in the example above about the sum of the first n odd numbers. However, you could use proof by exhaustion to prove that 101 is prime, since you could test to see if 101 is divisible by any number less than 101.
- Proof by deduction: this is when you use known results to deduce further results. For example, there are several ways to prove Pythagoras' theorem, using results you already know, such as the area of a triangle.
- Proof by contradiction: here you assume that the result is *not* true, and use this assumption to deduce a result which is impossible, or contradicts the original assumption. This means that the original assumption must be wrong.

Proof by induction

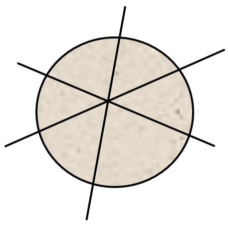
In this section you meet another method of proof, called proof by induction. Proof by induction is a topic that many students find difficult. In fact it is not really all that hard to actually do the questions; the problem is in understanding how and why proof by induction works.

Here is a practical example which may help.

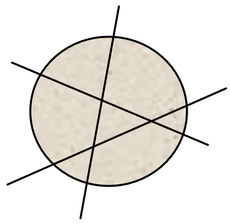
The problem: prove that the maximum number of pieces into which you can cut a pizza with n cuts is given by $\frac{n^2+n+2}{2}$.

First you need to give some thought to the general principles of successful pizza cutting. If you want to get the maximum number of pieces with the minimum number of cuts, the first important thing is not to allow more than two cuts to meet at the same point.

For example, three cuts all meeting at the same point would give six pieces, but three cuts which do not meet at the same point give seven pieces.

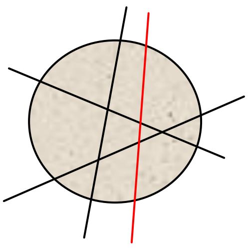


WRONG
6 pieces

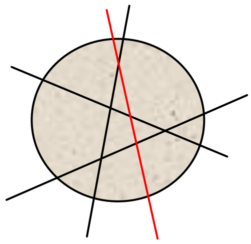


CORRECT
7 pieces

Secondly, you must make sure that each new cut you make crosses each of the previous cuts. The diagrams below show a fourth cut being added.



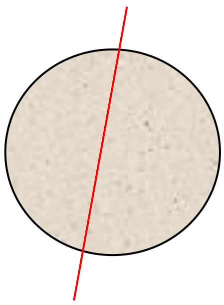
WRONG
10 pieces



CORRECT
11 pieces

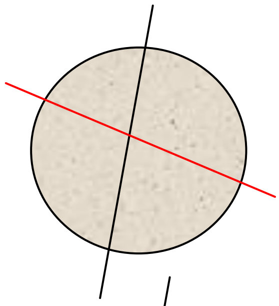
Let's now see how this works out for the first four cuts.

If you make 1 cut, you cut the pizza into two pieces.



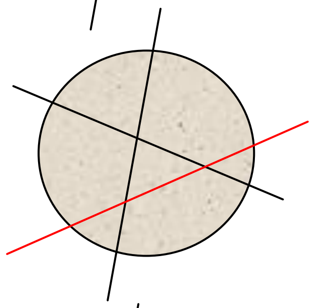
$n = 1$
2 pieces

When you make a second cut, you cut both pieces into two, giving two extra pieces, making a total of 4.



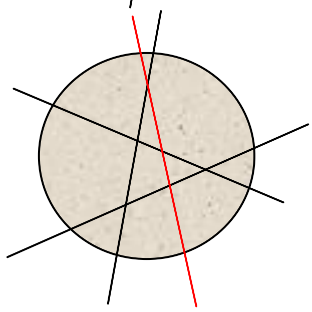
$n = 2$
4 pieces

When you make a third cut, you cross each of the two previous cuts. This means that you are cutting through three existing pieces, so you create three extra pieces.



$n = 3$
7 pieces

When you make a fourth cut, you cross each of the three previous cuts. This means that you are cutting through four existing pieces, so you create four extra pieces.



$n = 4$
11 pieces

You can see that the formula

$$\text{number of pieces from } n \text{ cuts} = \frac{n^2 + n + 2}{2}$$

works for the cases where $n = 1, 2, 3$ and 4.

In general, suppose that you already have k cuts and you want to add the $(k + 1)$th cut. You need to make sure that this cut crosses all k of the cuts you have made so far. This means that you are cutting through $k + 1$ pieces to create $k + 1$ new pieces. (You start off cutting a single piece into two, then cross a cut line, then cut a second piece in two, then cross a second cut line, etc., until you have crossed to the other side of the pizza, cutting the last piece you encounter in two. That's one more new piece made than the cuts crossed).

Now that you know how the pattern works, you can continue to check that the formula holds for different numbers of cuts without drawing the pizzas, just adding on from previous results.

You know that 4 cuts produces 11 pieces.

For 5 cuts, the number of pieces $= 11 + 5 = 16 = \dfrac{5^2 + 5 + 2}{2}$

so the formula is correct for $n = 5$.

For 6 cuts, the number of pieces $=16+6=22=\dfrac{6^2+6+2}{2}$
so the formula is correct for $n = 6$.

For 7 cuts, the number of pieces $=22+7=29=\dfrac{7^2+7+2}{2}$
so the formula is correct for $n = 7$.

Suppose you want to see if the formula is true for $n = 100$.
You could use the formula to work out the number of pieces for $n = 99$:

Number of pieces $=\dfrac{99^2+99+2}{2}=4951$.

Then you could use this result to find the result for $n = 100$.

Number of pieces $=4951+100=5051=\dfrac{100^2+100+2}{2}$.

So, the formula seems to work for $n = 100$, but to work this out you have *assumed* that the formula works for $n = 99$. All the above calculation tells you is that **IF** the formula is true for $n = 99$, **THEN** it is true for $n = 100$.

You could check $n = 99$ in the same way, by assuming that the formula works for $n = 98$, and showing that adding on 99 gives the correct result for $n = 99$. This now tells you that **IF** the formula is true for $n = 98$, **THEN** it is true for $n = 99$.

You could carry on working backwards like this, until you get down to a result which you already know is true, such as $n = 7$. Not a very efficient method of proof, and it doesn't prove that the result is true for **ALL** values of n. However, you can generalise this process to show that it is true for all values of n.

Assume that the formula is correct for the first k cuts. This means that you already have $\dfrac{k^2+k+2}{2}$ pieces. You want to show that for $k + 1$ cuts, the number of pieces is given by $\dfrac{(k+1)^2+(k+1)+2}{2}$, which can be simplified to $\dfrac{k^2+3k+4}{2}$.

So, after the $(k + 1)$th cut, which gives $k + 1$ additional pieces, the total number of pieces is given by

$$\frac{k^2+k+2}{2}+k+1$$

Simplifying gives $$\frac{k^2+k+2+2k+2}{2}=\frac{k^2+3k+4}{2}$$

which is the expected result for $k + 1$ cuts.

What you have now shown is that **IF** the formula is true for $n = k$, **THEN** it is true for $n = k + 1$.

This is true for **ANY** value of k.
So, in a few lines, you have shown that:
IF the formula is true for $n = 99$, **THEN** it is true for $n = 100$
IF the formula is true for $n = 42$, **THEN** it is true for $n = 43$
IF the formula is true for $n = 10$, **THEN** it is true for $n = 11$
IF the formula is true for $n = 13927$, **THEN** it is true for $n = 13928$
etc.

Of course this applies to **ALL** possible values!

This means that all you need to do is to check that the result is true for an initial case, say $n = 1$, and you can then say:

Since the formula is true for $n = 1$, then it must be true for $n = 2$.
Now since the formula is true for $n = 2$, then it must be true for $n = 3$.
Now since the formula is true for $n = 3$, then it must be true for $n = 4$.
And so on.

You can continue this as far as you like. So the formula is true for all values of $n \geq 1$.

There are three essential steps in a proof by induction:

Step 1 Prove that the result is true for a starting value, such as $n = 1$.
Step 2 Prove that if it is true for $n = k$, then it is true for $n = k + 1$.
Step 3 Conclude the argument.

Example 1

Prove that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$.

Solution

Step 1: When $n = 1$, $\sum_{r=1}^{1} r = 1$

When $n = 1$, $\frac{n(n+1)}{2} = \frac{1 \times 2}{2} = 1$

So it is true for $n = 1$.

Step 2: Assume that it is true for $n = \text{k}$, so $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$

$$\text{For } n = k+1,\ \sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= (k+1)\left(\frac{k}{2}+1\right)$$
$$= (k+1)\left(\frac{k+2}{2}\right)$$
$$= \frac{(k+1)((k+1)+1)}{2}$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers by induction.

Step 3 is just writing down the couple of sentences shown in the example above. You MUST include this: marks will be given for it. Remember that $n = 1$ may not always be the starting point!

For a challenge, click here and try to find the fallacy in the "proof by induction".

Other types of proof by induction

Proof by induction is often used to prove formulae for the sum of a series. However, there are many other situations in which it can be used.

Matrix powers

Example 2

For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ prove that $\mathbf{A}^n = \begin{pmatrix} 1 & 2^{n+1}-2 \\ 0 & 2^n \end{pmatrix}$

Solution

When $n=1$, $\mathbf{A}^1 = \begin{pmatrix} 1 & 2^2-2 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 4-2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = \mathbf{A}$

So the result is true for $n=1$

Assume $\mathbf{A}^k = \begin{pmatrix} 1 & 2^{k+1}-2 \\ 0 & 2^k \end{pmatrix}$

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$$
$$= \begin{pmatrix} 1 & 2^{k+1}-2 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2+2(2^{k+1}-2) \\ 0 & 2^k \times 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2+2^{k+2}-4 \\ 0 & 2^{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2^{k+2}-2 \\ 0 & 2^{k+1} \end{pmatrix}$$

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers by induction.

Sequences

Example 3
Given that $u_1 = 2$ and $u_{n+1} = 4u_n + 3$, prove that $u_n = 3\times 4^{n-1} - 1$

Solution
For $n = 1$, $u_1 = 3\times 4^{1-1} - 1 = 3 - 1 = 2$
So the result is true for $n = 1$

Assume that $u_k = 3\times 4^{k-1} - 1$
$$u_{k+1} = 4u_k + 3$$
$$= 4(3\times 4^{k-1} - 1) + 3$$
$$= 3\times 4^k - 4 + 3$$
$$= 3\times 4^k - 1$$

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers by induction.

Divisibility

Example 4
Prove that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all $n \geq 1$

Solution
For $n = 1$, $2^{n+2} + 3^{2n+1} = 2^3 + 3^3 = 8 + 27 = 35$ which is a multiple of 7.

Assume that $2^{k+2} + 3^{2k+1}$ is a multiple of 7 for some k.

So $2^{k+2} + 3^{2k+1} = 7m$ for some integer m
so $3^{2k+1} = 7m - 2^{k+2}$
For $n = k+1$, $2^{(k+1)+2} + 3^{2(k+1)+1} = 2^{k+3} + 3^{2k+3}$

$$= 2 \times 2^{k+2} + 9 \times 3^{2k+1}$$
$$= 2 \times 2^{k+2} + 9(7m - 2^{k+2})$$
$$= 2 \times 2^{k+2} + 63m - 9 \times 2^{k+2}$$
$$= 63m - 7 \times 2^{k+2}$$
$$= 7(9m - 2^{k+2})$$

which is a multiple of 7.

So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers by induction.

