Section 1: Summing series

Notes and Examples

These notes contain subsections on

- The sum of the first *n* natural numbers
- The sum of the squares of the first *n* natural numbers
- The sum of the cubes of the first *n* natural numbers

The sum of the first *n* natural numbers

The sum of the first *n* natural numbers, $1 + 2 + 3 + \ldots + n$ can be expressed in sigma notation as $\sum_{r=1}^{n} r$. The formula for this series is

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

This formula can be proved in many different ways. Here are two methods.

- **Proof of $\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$ using the sum of an arithmetic series**

  If you have covered arithmetic series (A level Mathematics), you will know that an arithmetic series is a series in which each term differs from the last by a fixed number, the common difference. You will also know that the sum of an arithmetic series with n terms is given by $S_n = \tfrac{1}{2}n[2a + (n-1)d]$, where a is the first term and d is the common difference.

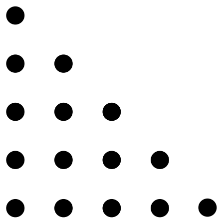
  $\sum_{r=1}^{n} r = 1 + 2 + 3 + ... + n$ is an arithmetic series with a = 1 and d = 1.

  So $\sum_{r=1}^{n} r = \tfrac{1}{2}n[2a + (n-1)d]$

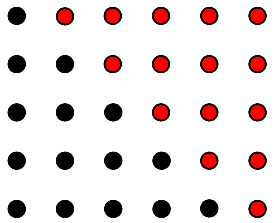
  $$\begin{aligned} &= \tfrac{1}{2}n[2 + (n-1)\times 1] \\ &= \tfrac{1}{2}n[2 + n - 1] \\ &= \tfrac{1}{2}n(n+1) \end{aligned}$$

- **Proof of $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ using triangle numbers**

  The nth triangle number, T_n, is the number of dots in a triangular array having n rows, with 1 dot on the top row, 2 dots on the second and so on, and n dots on the nth row. The sum $1 + 2 + 3 + \ldots + n$ is therefore T_n.



  The triangle for T_n can be put next to the same triangle drawn upside down, to form a rectangle with n rows and $n + 1$ columns.



  The number of dots in this rectangle is given by $n(n + 1)$, so the nth triangle number $T_n = \frac{1}{2}n(n+1)$

Example 1

Find (i) $\sum_{r=1}^{50} r$

(ii) $\sum_{r=50}^{100} r$

Solution

(i) $\sum_{r=1}^{50} r = \frac{1}{2}\times 50\times 51$

$= 1275$

(ii) $\sum_{r=50}^{100} r = \sum_{r=0}^{100} r - \sum_{r=0}^{49} r$

$= \frac{1}{2}\times 100\times 101 - \frac{1}{2}\times 49\times 50$

$= 3825$

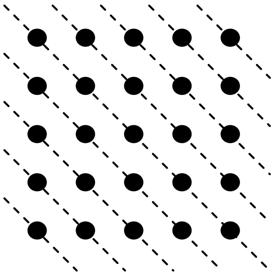
The sum of the squares of the first *n* natural numbers

The sum of the squares of the natural numbers is given by the formula

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

There are many ways to prove this formula. Here is one.

- **Proof of** $\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$



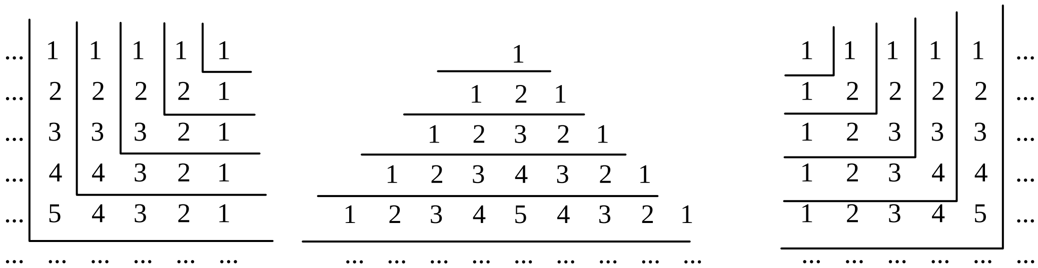
The diagram above shows that

$$r^2 = 1+2+3+\ldots+(r-1)+r+(r-1)+\ldots+3+2+1$$

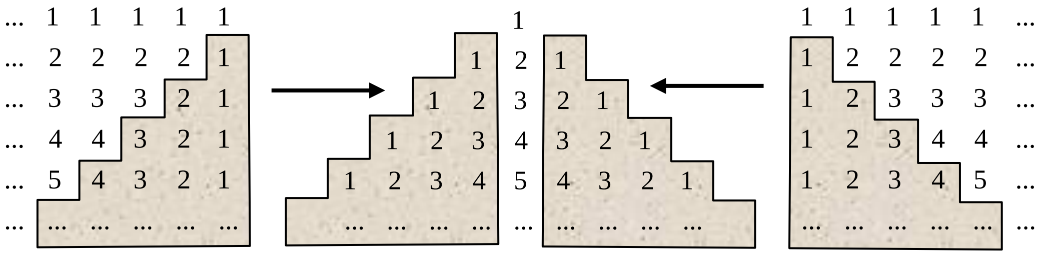
So $\sum_{r=1}^{n} r^2$ can be expressed as

$$\sum_{r=1}^{n} r^2 = 1+(1+2+1)+(1+2+3+2+1)+(1+2+3+4+3+2+1)+\ldots+(1+2+\ldots+r+\ldots+2+1)$$

So each of the three diagrams below represents $\sum_{r=1}^{n} r^2$.



Now imagine that you can slide the left and right diagrams on to the centre diagram so that the shaded boxes shown below overlap …



… and add the overlapping numbers:

...	1	1	1	1	1	1	1	1	1	...
...	2	2	2	2	2	2	2	2	2	...
...	3	3	3	3	3	3	3	3	3	...
...	4	4	4	4	4	4	4	4	4	...
...	5	5	5	5	5	5	5	5	5	...
...	...	...	...	...	...	...	...	...	...	...

The grid above has $2n + 1$ columns, and so it represents $(2n+1)\sum_{r=1}^{n} r$.

The original three grids each represent $\sum_{r=1}^{n} r^2$, so this gives

$$
\begin{aligned}
3\sum_{r=1}^{n} r^2 &= (2n+1)\sum_{r=1}^{n} r \\
&= (2n+1)\tfrac{1}{2}n(n+1) \\
&= \tfrac{1}{2}n(n+1)(2n+1)
\end{aligned}
$$

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

The sum of the cubes of the first *n* natural numbers

The sum of the cubes of the natural numbers is given by the formula

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Here is a geometrical proof.

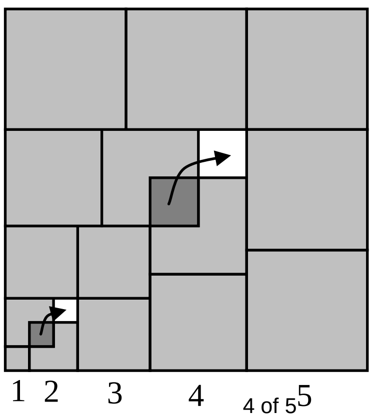
- **Proof of** $\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$

  Since $r^3 = r \times r^2$, then r^3 can be represented as the area of r squares of side r.

  This means that $\sum_{r=1}^{n} r^3$ is the total area of:

  1 square of side 1 + 2 squares of side 2 + 3 squares of side 3 + …. + n squares of side n.

  These can be arranged as shown below up to $n = 5$.



The even sided squares overlap, but there is a gap (unshaded) of exactly the same size as the overlap, so the overlapping regions can be moved to fill the gaps.
This shows that

$$\sum_{r=1}^{n} r^3 = (1+2+3+...+n)^2 = \left(\sum_{r=1}^{n} r\right)^2$$
$$= \left(\tfrac{1}{2}n(n+1)\right)^2$$
$$= \tfrac{1}{4}n^2(n+1)^2$$

Any series which can be expressed in the form $\sum_{r=1}^{n}(ar^3+br^2+cr+d)$ can be expressed as $\sum_{r=1}^{n}\left(ar^3+br^2+cr+d\right)=a\sum_{r=1}^{n}r^3+b\sum_{r=1}^{n}r^2+c\sum_{r=1}^{n}r+\sum_{r=1}^{n}d$ and summed using the standard results for the series $\sum_{r=1}^{n}r$, $\sum_{r=1}^{n}r^2$ and $\sum_{r=1}^{n}r^3$.

Note that $\sum_{r=1}^{n}d = d+d+d+...+d = nd$.

Example 2

Find $\sum_{r=1}^{n}(r^2+2r-1)$.

Solution

$$\sum_{r=1}^{n}(r^2+2r-1)=\sum_{r=1}^{n}r^2+2\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$$
$$=\tfrac{1}{6}n(n+1)(2n+1)+2\times\tfrac{1}{2}n(n+1)-n$$
$$=\tfrac{1}{6}n\left[(n+1)(2n+1)+6(n+1)-6\right]$$
$$=\tfrac{1}{6}n(2n^2+3n+1+6n+6-6)$$
$$=\tfrac{1}{6}n(2n^2+9n+1)$$

$\sum_{r=1}^{n}1=n$

Take out a factor $\tfrac{1}{6}n$