Section 2: Complex roots of polynomials

Exercise level 3

1. Which of the following can be a root of the equation $x^2 + ax + b = 0$ given that a and b are integers?

   $$\frac{1+\mathrm{i}}{4}, \quad \frac{1+\mathrm{i}\sqrt{11}}{2}, \quad \frac{1}{2}+\mathrm{i}, \quad \frac{1-\mathrm{i}\sqrt{7}}{2}.$$

2. Consider the equation

   $$z^3 - az^2 + az - 1 = 0$$

   where a is real.

   (i) Find a real root of the equation.

   (ii) Find the range of values of a such that the equation has complex roots.

   (iii) Show that all complex roots of the equation lie on the unit circle in the complex plane.

3. (i) Show that $(z+w)^* = z^* + w^*$ and $(zw)^* = z^* w^*$, where z and w are complex numbers.

   (ii) Let $\mathrm{q}(x) = a_m x^m$, where a_m is a real number. Show that $(\mathrm{q}(z))^* = \mathrm{q}(z^*)$ for all complex numbers z.

   (iii) Let $\mathrm{p}(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0$ are real numbers.

   Show that $(\mathrm{p}(z))^* = \mathrm{p}(z^*)$ for all complex numbers.

   Hence show that if $\mathrm{p}(z) = 0$ then $\mathrm{p}(z^*) = 0$.