Summary sheet: Differentiation

G1 Understand and use the derivative of f(x) as the gradient of the tangent to the graph of y = f(x) at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second derivatives; differentiation from first principles for small positive integer powers of x, Understand and use the second derivative as the rate of change of gradient
G2 Differentiate x^n, for rational values of n, and related constant multiples, sums and differences
G3 Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points, Identify where functions are increasing or decreasing

Notation:

There are a few ways of denoting differentiation. You might see all of the following:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \qquad y' \qquad \mathrm{f}'(x)$$

These are all ways of showing that something has been differentiated.

Gradients

- Differentiation is all about gradients. When you differentiate a function you have found the gradient function $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ which **<u>tells you the gradient at any point.</u>**

- Remember that the gradient of the curve is the same as the gradient of the tangent at that point.

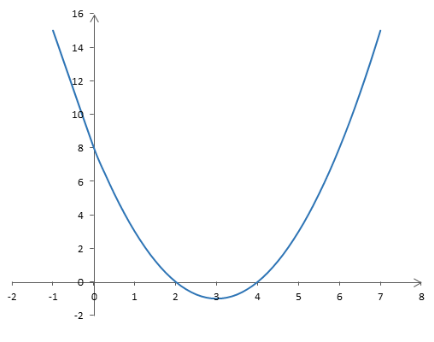
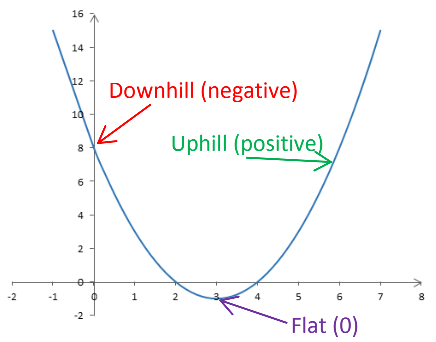
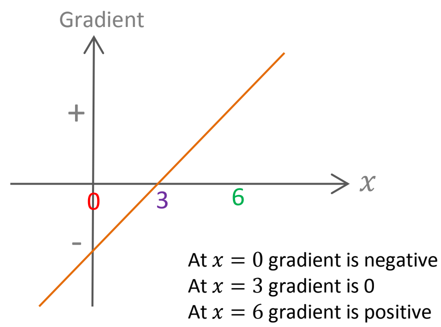
- If $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ the function in increasing (uphill) and if $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ the function in decreasing (downhill).

- The gradient function $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ measures the rate of change of y with respect to x $\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{x\to 0}\frac{\delta y}{\delta x}\right)$

- The gradient of a tangent at the point A on a curve = the limit of the gradient of the chord AP as P moves toward A along the curve. *(Imagine point A on a curve, you draw another point (P) and find the gradient of the line that joins them. You could move P closer and closer to A and the gradient would keep changing slightly, but the closer you get to A the more accurate the gradient is for point A).*

- To sketch the gradient function for a given curve, think about what is happening to the gradient at various points and sketch them.

e.g. Sketch the gradient of the given curve:

Curve	Think about the gradients	Gradient function
		

- If you differentiate a second time you have found the rate of change of the gradient – this is called the **<u>second derivative</u>** and can be used to find the type of turning point that you have. The second derivative is denoted $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or y'' or $\mathrm{f}''(x)$

Summary sheet: Differentiation

Differentiating from first principles

Remember that differentiation involves small changes in x (δx) and small changes in y (δy). So to differentiate from first principles just replace every x with $(x + \delta x)$ and every y with $(y + \delta y)$.

e.g. Differentiate $y = x^2 + 4x$ from first principles.

$$y = x^2 + 4x$$

Add the small changes

$$y + \delta y = (x + \delta x)^2 + 4(x + \delta x) \qquad (1)$$

Original

$$y = x^2 + 4x \qquad (2)$$

Subtract (1-2) to leave δy on its own

$$\delta y = (x + \delta x)^2 + 4(x + \delta x) - x^2 - 4x$$

Divide by δx to get $\frac{\delta y}{\delta x}$

$$\frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 + 4(x + \delta x) - x^2 - 4x}{\delta x}$$

Now tidy up & simplify

$$= \frac{x^2 + 2x\delta x + (\delta x)^2 + 4x + 4\delta x - x^2 - 4x}{\delta x}$$

$$= \frac{2x\delta x + (\delta x)^2 + 4\delta x}{\delta x}$$

$$= 2x + \delta x + 4$$

So as $\delta x \to 0$ we have

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx} = 2x + 4$$

Differentiating

The good news is that there is a much easier way to differentiate without using first principles.

Original Function: $\boldsymbol{y = kx^n}$

Gradient function: $\boldsymbol{\frac{dy}{dx} = knx^{n-1}}$

Multiply the expression by the power then reduce the power by 1.

For functions with more than one term, differentiate each term separately. Remember for each term: **expression x power**, then **power -1**.

e.g. **Differentiate $y = 5x^3 - 3x^2 + 7x - 5$**

$$\frac{dy}{dx} = 15x^2 - 6x + 7$$

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Finding a gradient

Once you have found the gradient function you can use it to find the gradient at any point.

e.g. Find the gradient of the curve $y = 5x^3 - 3x^2 + 7x - 5$ at the point $x = 3$

We already know that the gradient function is:

$$\frac{dy}{dx} = 15x^2 - 6x + 7$$

So the gradient when $x = 3$ can be found by:

$$\begin{aligned}\frac{dy}{dx} &= 15(3)^2 - 6(3) + 7 \\ &= \mathbf{124}\end{aligned}$$

Tangent and Normal

Remember that a **tangent** to a curve will have the **same** gradient and the **normal** will have a **perpendicular** gradient. So to find either of them you will need to start off with finding the gradient of the curve at the point you are interested in. Remember that both the tangent and the normal are straight lines and so they will be of the form $y = mx + c$ and you will need to find m (gradient) and c.

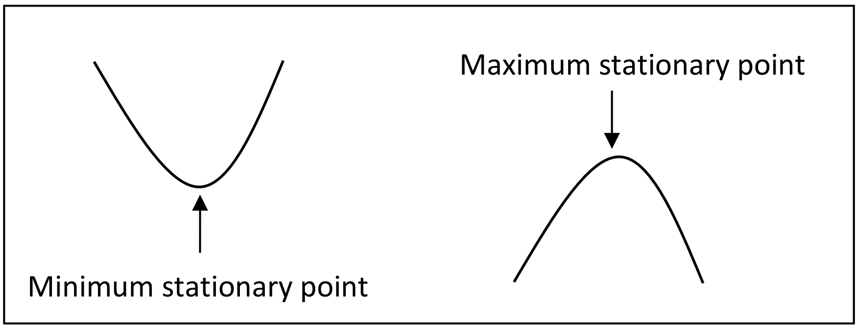
e.g. Find the tangent and the normal to the curve $y = 3x^2 - 4x + 2$ at the point $x = 2$

$$\frac{dy}{dx} = 6x - 4$$

So when $x = 2$ the gradient $= 6(2) - 4 = \mathbf{8}$
We also know that when $x = 2,\ y = 3(2)^2 - 4(2) + 2 = 6$

	Gradient	Equation	Substitute point $(2, 6)$ to find c	Equation
Tangent (same gradient)	8	$y = 8x + c$	$c = -10$	$y = 8x - 10$
Normal (perpendicular gradient)	$-\frac{1}{8}$	$y = -\frac{1}{8}x + c$	$c = \frac{25}{4}$	$y = -\frac{1}{8}x + \frac{25}{4}$

Stationary Points (maximum or minimum)



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Looking at the sketches you can see that at both the maximum and the minimum the gradient is 0. So to find where the stationary point is you need to find where the gradient=0. To decide what type of stationary point it is (max or min) you would use the second derivative (i.e. differentiate again).

1st derivative: tells you **where** the stationary point(s) is/are.

$\boxed{\text{Put } \frac{\mathrm{d}y}{\mathrm{d}x} = 0}$ and solve to find the value(s) of x.

Then substitute x into the original equation to find the y coordinate(s).

2nd derivative: tells you the **type** of stationary point(s) you have found.
Substitute your value(s) of x into the 2nd derivative and:

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\boxed{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0}$ it's a minimum (**positive** is **minimum**).

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\boxed{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0}$ it's a maximum (**negative** is **maximum**).

e.g. for the curve $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 11$ find the stationary point and decide whether it is a maximum or minimum.

$$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 11$$

Find **where** the stationary point(s) is(are):

Differentiate once: $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + x - 6$

Put $\frac{dy}{dx} = 0$: $x^2 + x - 6 = 0$

Solve (factorise or use quadratic formula) to get $\boldsymbol{x = 2}$ or $\boldsymbol{x = -3}$

Find **type** of stationary points:

Differentiate again: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x + 1$

When $x = 2$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2(2) + 1 = 5$ Positive so ***Minimum***

When $x = -3$: $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2(-3) + 1 = -5$ Negative so ***Maximum***