

Summary sheet: The binomial expansion

D1 Understand and use the binomial expansion of $(a + bx)^n$ for positive integer n ; the notations $n!$ and nCr ; link to binomial probabilities

The binomial expansion

The binomial expansion can be used to multiply out a bracket, to a power. Expanding $(x + y)^2$ would be quite easy to do as you would just multiply out the brackets. However if you were asked to expand $(x + y)^9$ it would be a long and tedious process, and so the binomial expansion helps you to do this.

To find the coefficients of the binomial expansion you can use **Pascal's Triangle**:

			1					
Row 1			1	1				
Row 2			1	2	1			
Row 3			1	3	3	1		
Row 4			1	4	6	4	1	
Row 5			1	5	10	10	5	1

Method:

- Use Pascal's triangle to find the coefficients
- The **1st term** in the brackets starts with the power of n and **decreases** to 0
- The **2nd term** in the brackets starts with the power of 0 and **increases** to n

Remember that anything to the power of zero is 1.

e.g. Expand $(a + b)^4$

Row 4 of Pascal's triangle gives the coefficients:	1	4	6	4	1
1 st term (a) power decreases:	a^4	a^3	a^2	a^1	a^0
2 nd term (b) power increases:	b^0	b^1	b^2	b^3	b^4

Expansion becomes: $1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

Tidies up to: $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ (you don't really need the 1s)

As a check that everything is correct, notice that the powers always add up to 4 (in each term).

If the terms are negative and/or have coefficients, the process is exactly the same but you need to be careful and make sure that you apply the power to the whole term. The safest way is to use brackets.

e.g. Expand $(3x - 2y)^4$

Row 4 of Pascal's triangle gives the coefficients:	1	4	6	4	1
1 st term ($3x$) power decreases:	$(3x)^4$	$(3x)^3$	$(3x)^2$	$(3x)^1$	$(3x)^0$
2 nd term ($-2y$) power increases:	$(-2y)^0$	$(-2y)^1$	$(-2y)^2$	$(-2y)^3$	$(-2y)^4$

Expansion becomes: $1(3x)^4 + 4(3x)^3(-2y) + 6(3x)^2(-2y)^2 + 4(3x)(-2y)^3 + 1(-2y)^4$

Tidies up to: $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$

Summary sheet: The binomial expansion

Finding the coefficients

You can also calculate the binomial coefficients without using Pascal's triangle. Remember that $n!$ (n factorial) means $n \times (n - 1) \times (n - 2) \dots \times 1$. For example $4! = 4 \times 3 \times 2 \times 1 = 24$.

You can calculate the coefficients using the following definition:

The power you are expanding to. \nearrow

The power of the coefficient you are looking for. \nearrow

$${}^n C_r = \frac{n(n-1)\dots(n-r+1)}{1 \times 2 \times 3 \times \dots \times r}$$

Stop at the difference between n and $r + 1$.

\swarrow $7 - 2 = 5$, so stop at 6

e.g. ${}^7 C_2 = \frac{7 \times 6}{1 \times 2} = \frac{42}{2} = 21$

e.g. for the expansion $(4 + 3x)^9$ find the coefficient of x^4

Think: you are expanding to the power of 9 and looking for the coefficient of x^4 so you will need ${}^9 C_4$
 You are looking for the coefficient of x^4 so you need the 2nd term to the power of 4, i.e. $(3x)^4$
 The powers will always add to 9 so the other term must be to the power of 5, i.e. 4^5

$$x^4 = {}^9 C_4 \times 4^5 \times (3x)^4 = \frac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} \times 4^5 \times 3^4 x^4 = 10450955x^4$$

The symmetry

The binomial coefficients are symmetrical, i.e. ${}^n C_r = {}^n C_{n-r}$ this means you can choose to do whichever calculation looks easier (e.g. ${}^7 C_5$ is the same as ${}^7 C_2$)

e.g. ${}^7 C_5 = \frac{7 \times 6 \times 5 \times 4 \times 3}{1 \times 2 \times 3 \times 4 \times 5} = \frac{42}{2} = 21$ (Remember previous e.g.: ${}^7 C_2 = \frac{7 \times 6}{1 \times 2} = \frac{42}{2} = 21$)

You can see that both ways give the same answer but with ${}^7 C_5$ there are more calculations (and some cancelling) to do before reaching the answer. So if you needed to find ${}^7 C_5$ you could find ${}^7 C_2$ instead.

Binomial probabilities

A binomial experiment consists of repeated trials (e.g. tossing a coin) where there are 2 outcomes (e.g. heads or tails) classed as a success or a failure. The combination ${}^n C_r$ is used in calculating probabilities.

Probability of X successes:

$$P(X) = {}^n C_x p^x q^{n-x}$$

Where: n = number of trials
 x = number of successes
 p = probability of success
 q = probability of failure (i.e. $(1 - p)$)

e.g. if you throw a die 8 times what is the probability of throwing three 6s?

$n = 8$

$x = 3$ (we want 3 6s)

$p = \frac{1}{6}$ (probability of throwing a 6)

$q = \frac{5}{6}$ (probability of NOT throwing a 6)

$P(3 \text{ 6s}) = {}^8 C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^5 = 0.104$