Summary sheet: Polynomials

B6 Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem
Sketch polynomial graphs
Simplify rational expressions including by factorising and cancelling, and algebraic division (by linear expressions only)

Adding & subtracting polynomials

Remember that you can only **collect "like terms"** together so if terms have different powers then you cannot add or subtract them.

e.g. If $\mathrm{f}(x) = 5x^3 + 2x^2 - 3x + 7$ **and** $\mathrm{g}(x) = 3x^3 - 4x^2 + 8x - 5$

Find $\mathrm{f}(x) + \mathrm{g}(x)$

$$(5x^3 + 2x^2 - 3x + 7) + (3x^3 - 4x^2 + 8x - 5) = 8x^3 - 2x^2 + 5x + 2$$

Add the x^3 terms (shown in red) then the x^2 terms (blue) then the x terms (green) and finally the numbers (black). For subtraction follow the same procedure but be careful with the signs. Remember that you have to subtract everything in the second bracket.

Multiplying polynomials (expanding brackets)

Remember to **multiply every term in the first bracket by every term in the second bracket**.

e.g. Multiply $(3x^2 - 4x + 7)$ **by** $(x^2 + 3x - 5)$

	$(3x^2 - 4x + 7)(x^2 + 3x - 5)$
Each term in 1st bracket X 2nd bracket	$= 3x^2(x^2 + 3x - 5) - 4x(x^2 + 3x - 5) + 7(x^2 + 3x - 5)$
Multiply each bracket out	$= 3x^4 + 9x^3 - 15x^2 - 4x^3 - 12x^2 + 20x + 7x^2 + 21x - 35$
Collect like terms together	$= 3x^4 + 9x^3 - 4x^3 - 15x^2 - 12x^2 + 7x^2 + 20x + 21x - 35$
Tidy up	$= 3x^4 + 5x^3 - 20x^2 + 41x - 35$

If there are 3 brackets (or more) you multiply the first 2 brackets then multiply the answer by the third bracket and continue until all the brackets have been multiplied out.

Algebraic division

There are different ways of doing algebraic division, so choose the one you prefer. The following examples show the methods of: 'equating the coefficients' and 'long division'.

You are dividing a cubic expression by a linear, so you know that the answer (quotient) will be a quadratic $\left(\frac{x^3}{x} = x^2\right)$

e.g. Divide $3x^3 + 4x^2 - 13x + 6$ **by** $x + 3$

By inspection:
If you know that there is no remainder, you can write he polynomial in factorised form 'by inspection'

$(x + 3)(3x^2 + \cdots x + 2)$ — The term in x^2 in the second bracket must be $3x^2$ to give the $3x^3$ term, and the constant term in the second bracket must be 2 to give the term 6.

$(x + 3)(3x^2 - 5x + 2)$ — Think about the x^2 term. $3 \times 3x^2 = 9x^2$, so to get the $4x^2$ term, you need the middle term to be $-5x$, as $x \times -5x = -5x^2$. Check that this also gives the correct x term.

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By long division:

$$
\begin{array}{rl}
 & 3x^2 - 5x + 2 \\
x + 3 & \overline{)\,3x^3 + 4x^2 - 13x + 6} \\
 & \underline{3x^3 + 9x^2} \\
 & \quad -5x^2 - 13x \\
 & \quad \underline{-5x^2 - 15x} \\
 & \qquad\qquad 2x + 6 \\
 & \qquad\qquad \underline{2x + 6} \\
 & \qquad\qquad\quad 0
\end{array}
$$

- Divide 1st term by x and put the answer above the answer line.
  $\left(\frac{3x^3}{x} = 3x^2\right)$
- Multiply $(x+3)$ by $3x^2$ $(= 3x^3 + 9x^2)$
- Place under the original and subtract $(= -5x^2)$
- Bring the $-13x$ down.
- Repeat the process.

By equating coefficients:

Let the quotient $= ax^2 + bx + c$ and any remainder $= d$

$$
\begin{aligned}
3x^3 + 4x^2 - 13x + 6 &= (x+3)(ax^2 + bx + c) + d \\
&= x(ax^2 + bx + c) + 3(ax^2 + bx + c) + d \\
&= ax^3 + bx^2 + cx + 3ax^2 + 3bx + 3c + d \\
3x^3 + 4x^2 - 13x + 6 &= ax^3 + (b + 3a)x^2 + (c + 3b)x + 3c + d
\end{aligned}
$$

After you have tidied everything up (as above) equate the coefficients to find the value of a, b, c and d.

Equate coefficients of x^3:	$\mathbf{3 = a}$	$\mathbf{a = 3}$
Equate coefficients of x^2:	$4 = b + 3a$	$\mathbf{b = -5}$
Equate coefficients of x:	$-13 = c + 3b$	$\mathbf{c = 2}$
Equate constants:	$6 = 3c + d$	$\mathbf{d = 0}$

You have found that: $3x^3 + 4x^2 - 13x + 6 \div (x+3) = 3x^2 - 5x + 2$

The factor theorem

This means that if you substitute a value of x into the equation and the answer is 0 you have found a factor and a root.

If $\mathrm{f}(a) = 0 \iff (x - a)$ is a factor of $\mathrm{f}(x)$
Also $x = a$ is a root of the equation $\mathrm{f}(x) = 0$

e.g. solve $x^3 + x^2 - 11x + 10 = 0$

First of all find a root by trial and error (i.e. try different numbers until you get the answer 0). Use the constant to decide what numbers to try. In this case the solution will have to be a factor of 10.

$\mathrm{f}(1) = 1$ $\quad$ $\mathrm{f}(2) = 0$ $\quad$ $\mathrm{f}(5) = 105$ $\quad$ $\mathrm{f}(10) = 1000$

You have found that 2 is a root and so $(x - 2)$ must be a factor. Now you can divide as shown above.

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Sketch polynomial graphs

To be able to sketch a graph you need to know:

- The **shape**
- the x and y **intercepts** (where it crosses the axes)
- where there are any the **turning point(s)**

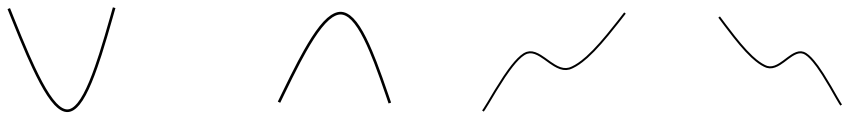
The turning point is sometimes called the stationary point or the maximum or minimum.

Also remember: a polynomial of order *n* meets the x-axis *n* times at most
a polynomial of order *n* has *n* – 1 turning points at most

If you're not sure what a graph looks like you can always find a few points and plot them until you have an idea of the shape.

Some general shapes to remember:

$y = x^2$ $y = -x^2$ $y = x^3 + \cdots$ $y = -x^3 + \cdots$



The intercepts

Remember: to find the $\boldsymbol{y}$**-intercept** substitute $\boldsymbol{x = 0}$ and solve
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The turning point

To find the turning point of a quadratic graph you can complete the square.

Simplify rational expressions

Rational expressions are fractions where the numerator and denominator are polynomials. Remember that you can only **cancel factors** and **not terms**.

$$\frac{(3x^2 + 2x) \times \cancel{5}}{2x \times \cancel{5}} \quad ✓ \qquad\qquad \frac{3x^2 + \cancel{2x + 5}}{\cancel{2x + 5}} \quad \text{X}$$

$$\frac{\cancel{(x+4)}(x-5)}{x\cancel{(x+4)}} \quad ✓ \qquad\qquad \frac{\cancel{(x+4)} + (x-5)}{x\cancel{(x+4)}} \quad \text{X}$$