Section 1: Trigonometric functions and identities

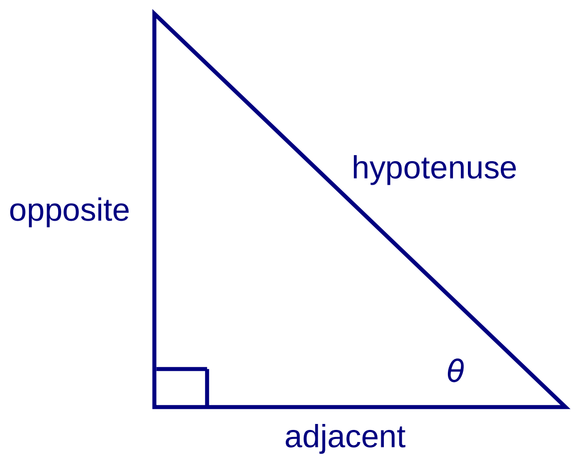
Notes and Examples

In this section you learn about the trigonometric functions and some trigonometric identities.

These notes contain sub-sections on:

The trigonometric functions for angles between 0° and 90°

From GCSE, you know that for a right-angled triangle the trigonometric functions are defined as:



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

You can also write: $\sin(90° - \theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ and $\cos(90° - \theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$

So: $\cos\theta = \sin(90° - \theta)$ and $\sin\theta = \cos(90° - \theta)$

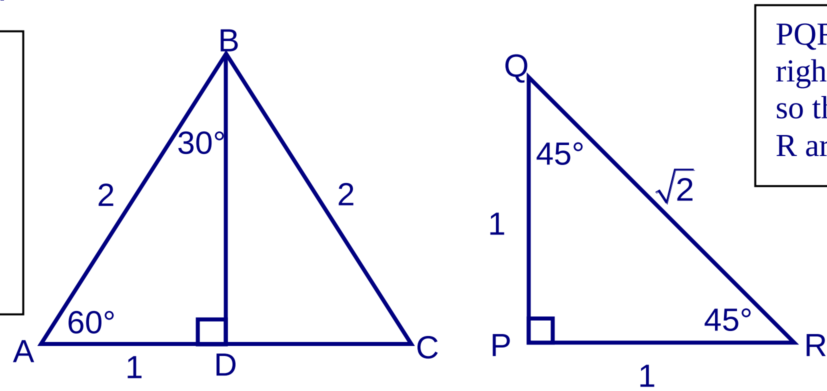
For example:
$\cos 70° = \sin 20°$
$\sin 15° = \cos 75°$

Edexcel Maths Trigonometry 1 Notes and Examples

Common values of sin *θ*, cos *θ* and tan *θ*

The two triangles below help you to find the values of sin θ, cos θ and tan θ for θ = 30°, 45°, 60°.

ABC is an equilateral triangle, so all its angles are 60°. D is the midpoint of AC, so that triangle ABD is a right-angled triangle with AD = 1.



PQR is an isosceles right-angled triangles, so the angles at Q and R are 45°.

You should learn the values of sin θ, cos θ and tan θ for θ = 0°, 30°, 45°, 60° and 90°. These are shown in the table below.

θ	0°	30°	45°	60°	90°
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan θ	0	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

In the example below you need to substitute exact values into an expression.

Example 1

Show that $\sin^2 30° + \cos^2 30° = 1$

Solution

$\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$

So substituting these values into $\sin^2 30° + \cos^2 30°$:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$ as required.

Edexcel Maths Trigonometry 1 Notes and Examples

Trigonometric functions for angles of any size

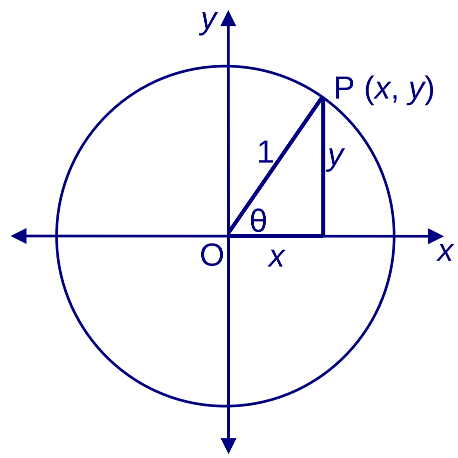
The definitions for sine, cosine and tangent can be extended to angles of any size using a diagram like the one below.

This gives the definitions:

$$\sin\theta = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$



Think about the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$ from the definitions above.

- In the first quadrant, θ lies between 0° and 90°. The values of x and y are both positive, so the values of sin θ, cos θ and tan θ are all positive.
- In the second quadrant, θ lies between 90° and 180°. The value of x is negative and the value of y is positive, so sin θ is positive but cos θ and tan θ are both negative.
- In the third quadrant, θ lies between 180° and 270° (or alternatively, between -90° and -180°). The values of x and y are both negative, so tan θ is positive but sin θ and cos θ are both negative.
- In the fourth quadrant, θ lies between 270° and 360° (or alternatively, between 0° and -90°). The values of x is positive and the value of y is negtive, so cos θ is positive but sin θ and tan θ are both negative.

This can be summarised in the diagram below.

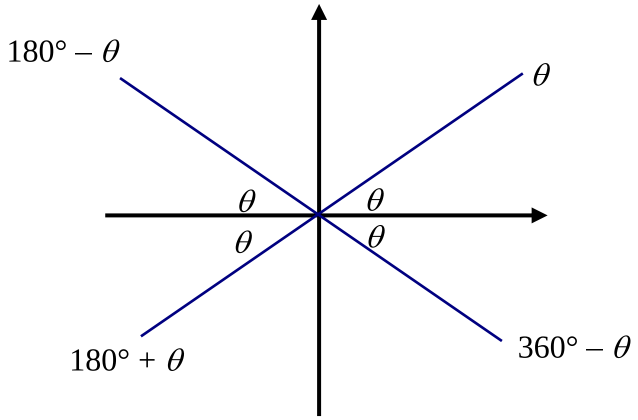
Sine positive	All positive
Tangent positive	Cosine positive

or, more simply

S	A
T	C

This is often known as the “CAST” diagram: some people remember it using the word **CAST** (make sure you start the word in the right place!); others use a phrase such as “**A**ll **S**chool **T**eachers **C**riticise” (which, while rather unfair to the teaching

profession, does have the advantage of starting in the first quadrant).
The symmetry of the diagram allows you to find relate trig ratios of angles in the second, third and fourth quadrant to trig ratios angles in the first quadrant.

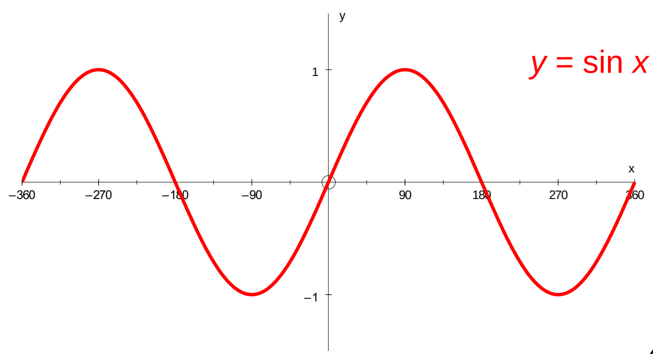


To find a trig ratio of an angle in the second, third or fourth quadrant in terms of the trig ratio of an acute angle, you need to find the equivalent acute angle using symmetry, and then use the CAST diagram to find whether the trig ratio is positive or negative.

Graphs of trigonometric functions

Now that you can define sin θ, cos θ and tan θ for any value of θ you can draw the graphs of these functions:

Graph of *y* = sin *x*



e.g. sin 15° = sin 375°
and sin 15° = sin -345°

Note:

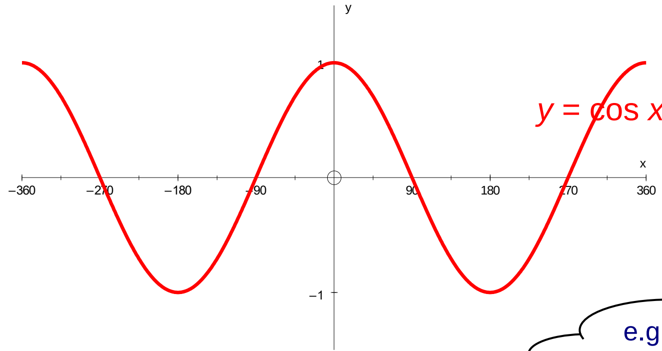
1. The graph of $y = \sin x$ has a period of 360°.
   So it repeats every 360°.
2. It has rotational symmetry about the origin.
3. $-1 \leq \sin x \leq 1$
   So $y = \sin x$ lies between 1 and -1.
4. There is a line of symmetry at $x = 90°$ and $x = -90°$.

e.g. sin 30°= 0.5
so sin -30° = -0.5

e.g. sin *x* = sin (180° - *x*)

Edexcel Maths Trigonometry 1 Notes and Examples

Graph of *y* = cos *x*

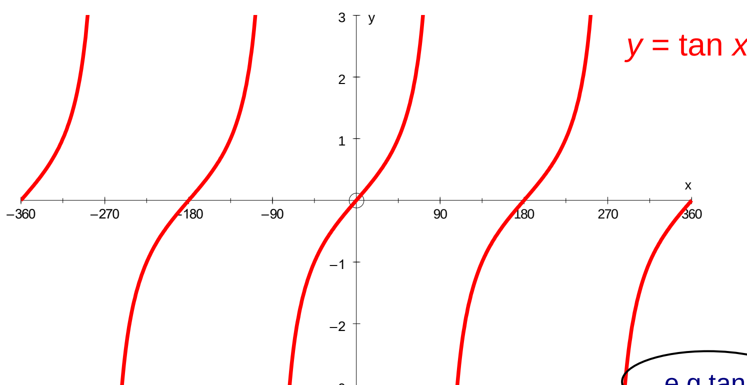


e.g. cos 25° = cos 385°
and cos 25° = cos -335°

Note:

1. The graph of $y = \cos x$ has a period of 360°.
   So it repeats every 360°.
2. It has a line of symmetry in the *y*-axis.
   e.g. cos 30°= cos -30°
3. $-1 \le \cos x \le 1$
   So $y = \cos x$ lies between 1 and -1.
4. The graph of $y = \cos x$ is a translation of the graph $y = \sin x$ by 90° to the left.
   In general, $\cos\theta \equiv \sin(\theta + 90°)$
   e.g. cos 0° = sin 90°
   and cos 10° = sin 100°

Graph of *y* = tan*x*



e.g.tan 15° = tan 195°
and tan 15° = tan -165°

Note:

1. The graph of $y = \tan x$ has a period of 180°.
   So it repeats every 180°.
   e.g. tan 60°= $\sqrt{3}$
   so tan -60° = - $\sqrt{3}$
2. It has rotational symmetry about the origin.
3. $-\infty \le \tan x \le \infty$
   So tan *x* can be any value not just those between -1 and 1.
4. There are asymptotes at $x = \pm 90°$, $x = \pm 270°$, $x = \pm 450°$ … where the value of $\tan x$ is undefined (since $\cos x = 0$ for these values of *x*)

In the next example you need to use the symmetries of the trigonometric graphs.

Example 2
Write down the exact values of
(i) tan 120°
(ii) sin (-135°)
(iii) cos 300°

'Exact' means leave as a fraction or a surd – don't just give all the decimal places from your calculator!

Solution
(i) $y = \tan\theta$ has period of 180° so tan 120° tan (120° - 180°) = tan (-60°).
$y = \tan\theta$ has rotational symmetry about the origin so tan (-60°) = -tan 60° $= -\sqrt{3}$
So $\tan 120° = -\sqrt{3}$

(ii) $y = \sin\theta$ has rotational symmetry so sin (-135°) = -sin 135°.
$y = \sin\theta$ has a line of symmetry at $\theta = 90°$ so sin 135° = sin (180° - 135°) = sin 45°

So sin (-135°) = -sin 45° $= -\frac{1}{\sqrt{2}}$

(iii) $y = \cos\theta$ has period of 360° so cos 300° = cos (300° - 360°) = cos (-60°).
$y = \cos\theta$ is symmetrical about the y-axis so cos (-60°) = cos 60°.
So cos 300° = cos 60° = $\frac{1}{2}$.

Trigonometric identities

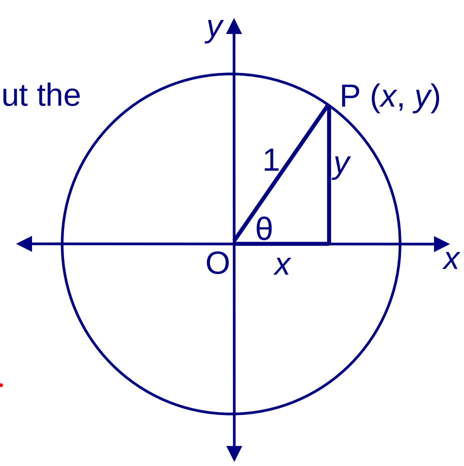
You need to know the following identities:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta \equiv 1$$

e.g. $\tan 30° = \frac{\sin 30°}{\cos 30°}$

An identity is true for all values of θ.

You can prove both these identities by thinking about the circle diagram used earlier.



Since $\cos\theta = x$, $\sin\theta = y$ and $\tan\theta = \frac{y}{x}$,

you can see immediately that $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$,

and Pythagoras' theorem leads to $\sin^2\theta + \cos^2\theta \equiv 1$.

In the next example you need to use the trigonometric identities to rewrite an expression.

Example 3
Show that $(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)=2\sin^2\theta-1$

Solution
Working with the LHS and expanding the brackets gives:

$$(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)=\sin^2\theta-\cos^2\theta \qquad ①$$

Since $\sin^2\theta+\cos^2\theta\equiv 1$ then $\cos^2\theta\equiv 1-\sin^2\theta$ ②

Substituting ② into ① gives:

$$(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)=\sin^2\theta-(1-\sin^2\theta)$$

Simplifying: $(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)=2\sin^2\theta-1$ as required.