Section 1: Surds

Notes and Examples

These notes contain subsections on

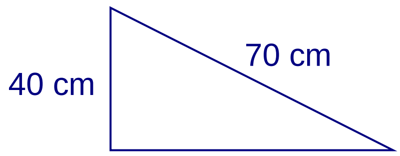
- Rational and irrational numbers
- Writing a square root in terms of a simpler square root
- Adding and subtracting surds
- Multiplying surds
- Rationalising the denominator

Remember, an **irrational number** is a number which cannot be expressed as one whole number divided by another whole number.

Rational and irrational numbers

The square root of any number which is not itself a perfect square is an **irrational number**. So $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers, but $\sqrt{4}$ is not as it is equal to 2, which is a rational number. A number which is partly rational and partly square root (or cube root etc.) is called a **surd**. (There are of course other irrational numbers which do not involve a root, such as π.

In this section you will learn to manipulate and simplify expressions involving surds. This is an important skill in many areas of mathematics. For example, suppose you have a triangular paving slab like this:



You can use Pythagoras' theorem to work out that the length of the third side is $\sqrt{3300}$. You could use a calculator to work this out and give the answer to two or three decimal places, but this would no longer be exact. Suppose you wanted to find the area and perimeter of the slab, or the total area of 100 slabs, or find out how many slabs you could make from a certain volume of concrete? It is much better to use the exact answer in these calculations, and then the results will also be exact.

Writing a square root in terms of a simpler square root

Square roots like the one in the example above look quite daunting and can be difficult to work with. However, many square roots can be written in terms of a simpler square root like $\sqrt{2}$ or $\sqrt{3}$ (and the same applies to cube roots and so on). Example 1 shows how to do this.

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Example 1
Write these numbers in terms of the simplest possible square roots.
(i) $\sqrt{12}$ (ii) $\sqrt{72}$ (iii) $\sqrt{150}$

Solution
(i) $12 = 4 \times 3$
$$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$$

Look for any factors which are perfect squares

(ii) $72 = 36 \times 2$
$$\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$$

(iii) $150 = 25 \times 6$
$$\sqrt{150} = \sqrt{25} \times \sqrt{6} = 5\sqrt{6}$$

Adding and subtracting surds

Adding and subtracting surds is rather like adding or subtracting algebraic expressions, in that you have to collect “like terms”. You should collect together any rational numbers, and collect together any terms involving roots of the same number. You cannot collect together terms involving roots of different numbers, such as $\sqrt{2}$ and $\sqrt{3}$.

Example 2
Simplify
(i) $(2+\sqrt{2})+(3-2\sqrt{2})$
(ii) $(4-\sqrt{3})-(1-2\sqrt{2}+3\sqrt{3})$
(iii) $\sqrt{32}-\sqrt{18}$

Solution
(i)
$$\begin{aligned}(2+\sqrt{2})+(3-2\sqrt{2}) &= 2+3+\sqrt{2}-2\sqrt{2}\\ &= 5-\sqrt{2}\end{aligned}$$

(ii)
$$\begin{aligned}(4-\sqrt{3})-(1-2\sqrt{2}+3\sqrt{3}) &= 4-\sqrt{3}-1+2\sqrt{2}-3\sqrt{3}\\ &= 4-1-\sqrt{3}-3\sqrt{3}+2\sqrt{2}\\ &= 3-4\sqrt{3}+2\sqrt{2}\end{aligned}$$

(iii) $\sqrt{32}-\sqrt{18}$
$$\sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$$
$$\sqrt{18} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$$
$$\begin{aligned}\sqrt{32}-\sqrt{18} &= 4\sqrt{2}-3\sqrt{2}\\ &= \sqrt{2}\end{aligned}$$

At first sight this looks like it cannot be simplified. However, both surds can be written in terms of simpler surds (as in Example 1)

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For further practice in examples like the ones above, use the ***Simplifying surds skill pack***.

Multiplying surds

Multiplying two or more square roots is quite simple – you just multiply the numbers. You may then be able to write the result as a simpler surd.

Example 3
Simplify
(i) $\sqrt{2}\times\sqrt{6}$
(ii) $2\sqrt{15}\times\sqrt{6}\times3\sqrt{10}$

Solution
(i) $\sqrt{2}\times\sqrt{6}=\sqrt{12}$
$$=\sqrt{4}\times\sqrt{3}$$
$$=2\sqrt{3}$$

The approach used in (i) works well for small numbers, but for (ii) you would get the square root of a large number to simplify. An easier way for this example is to split each surd into simpler ones and then look for any pairs.

(ii) $2\sqrt{15}\times\sqrt{6}\times3\sqrt{10}=(2\times\sqrt{3}\times\sqrt{5})\times(\sqrt{2}\times\sqrt{3})\times(3\times\sqrt{5}\times\sqrt{2})$
$$=2\times3\times\sqrt{3}\times\sqrt{3}\times\sqrt{5}\times\sqrt{5}\times\sqrt{2}\times\sqrt{2}$$
$$=6\times3\times5\times2$$
$$=180$$

The next example deals with multiplying expressions involving a mixture of rational numbers and roots. You have to use brackets for this, and it is very similar to multiplying out two brackets in algebra – each term in the first bracket needs to be multiplied by each term in the second bracket. You can use FOIL (First, Outer, Inner, Last) if it helps you.

Example 4
Multiply out and simplify
(i) $(2+\sqrt{3})(1-2\sqrt{3})$

(ii) $(3-\sqrt{2})^2$

(iii) $(\sqrt{5}-2)(\sqrt{5}+2)$

Solution

(i) $$\begin{aligned}(2+\sqrt{3})(1-2\sqrt{3}) &= 2-4\sqrt{3}+\sqrt{3}-2\sqrt{3}\times\sqrt{3}\\ &= 2-3\sqrt{3}-6\\ &= -4-3\sqrt{3}\end{aligned}$$

(ii) $$\begin{aligned}(3-\sqrt{2})^2 &= (3-\sqrt{2})(3-\sqrt{2})\\ &= 9-3\sqrt{2}-3\sqrt{2}+\sqrt{2}\times\sqrt{2}\\ &= 9-6\sqrt{2}+2\\ &= 11-6\sqrt{2}\end{aligned}$$

(iii) $$\begin{aligned}(\sqrt{5}-2)(\sqrt{5}+2) &= \sqrt{5}\times\sqrt{5}+2\sqrt{5}-2\sqrt{5}-2\times 2\\ &= 5-4\\ &= 1\end{aligned}$$

For further practice in examples like the one above, try the ***Multiplying surds skill pack***.

Part (iii) of Example 4 illustrates a very important and useful idea. Multiplying out any expression of the form $(a+b)(a-b)$ gives the result a^2-b^2. The "outer" and "inner" products, $-ab$ and ab, cancel each other out. When either or both of a and b are surds, the result a^2-b^2 is a rational number.

Rationalising the denominator

Surds in the denominator of a fraction can be a real nuisance! However, you can get rid of them from the denominator by a process called *rationalising the denominator*, which uses the idea above. If the denominator of a fraction is $a+b$, where either or both of a and b are surds, then you can multiply both top and bottom of the fraction by $a-b$. The denominator is then $(a+b)(a-b)$, which works out to be rational. *Since you multiply the top and bottom of the fraction by the same amount, its value is unchanged*. The numerator still involves surds, but this is not quite so difficult to work with.

This technique is shown in Example 5.

Example 5

Simplify the following by rationalising the denominator.

(i) $\dfrac{1}{\sqrt{3}}$ (ii) $\dfrac{1}{\sqrt{2}-1}$ (iii) $\dfrac{2+\sqrt{3}}{1-\sqrt{3}}$

Solution

(i) $$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{3}$$

Multiply top and bottom by $\sqrt{3}$

(ii) $$\frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$
$$= \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)}$$
$$= \frac{\sqrt{2}+1}{2-1}$$
$$= \sqrt{2}+1$$

Multiply top and bottom by $\sqrt{2}+1$

(iii) $$\frac{2+\sqrt{3}}{1-\sqrt{3}} = \frac{2+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$= \frac{(2+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$
$$= \frac{2+2\sqrt{3}+\sqrt{3}+3}{1-3}$$
$$= -\tfrac{1}{2}(5+3\sqrt{3})$$

Multiply top and bottom by $1+\sqrt{3}$

For further practice in examples like the one above, try the ***Rationalising the denominator skill pack***.