Section 1: Simultaneous equations

Notes and Examples

These notes contain subsections on

Linear simultaneous equations using elimination

This work is revision of GCSE. You need to make sure that you can solve linear simultaneous equations confidently before you move on to the work on one linear and one quadratic equation, which will probably be new to you.

Simultaneous equations involve more than one equation and more than one unknown. To solve them you need the same number of equations as there are unknowns.

One method of solving simultaneous equations involves adding or subtracting multiples of the two equations so that one unknown disappears. This method is called *elimination*, and is shown in the next example.

Example 1

Solve the simultaneous equations

$$3p + q = 5$$
$$p - 2q = 4$$

Solution

① $\quad 3p + q = 5$

② $\quad p - 2q = 4$

Adding or subtracting these equations will not eliminate either p or q. However, you can multiply the first equation by 2, and then add. This will eliminate p.

① × 2 $\quad 6p + 2q = 10$

② $\quad p - 2q = 4$

Adding: $\quad 7p = 14$

$$p = 2$$

$$3 \times 2 + q = 5$$
$$6 + q = 5$$
$$q = -1$$

Now substitute this value for p into one of the original equations – you can use either, but in this example, ① is used.

The solution is $p = 2$, $q = -1$.

Notice that, in Example 1, you could have multiplied equation ② by 3 and then subtracted. This would give the same answer.

Sometimes you need to multiply each equation by a different number before you can add or subtract. This is the case in the next example.

Example 2
5 pencils and 2 rubbers cost £1.50
8 pencils and 3 rubbers cost £2.35
Find the cost of a pencil and the cost of a rubber.

Solution

① $5p + 2r = 150$

② $8p + 3r = 235$

Let p represent the cost of a pencil and r represent the cost of a rubber. It is easier to work in pence.

① × 3 $15p + 6r = 450$

② × 2 $16p + 6r = 470$

The easiest method is to multiply equation ① by 3 and equation ② by 2. (You could of course multiply ① by 8 and ② by 5).

Subtracting: $-p = -20$

$p = 20$

$5 \times 20 + 2r = 150$

Substitute this value of p into equation ①

$100 + 2r = 150$

$2r = 50$

$r = 25$

A pencil costs 20p and a rubber costs 25p.

Linear simultaneous equations using substitution

An alternative method of solving simultaneous equations is called substitution. This can be the easier method to use in cases where one equation gives one of the variables in terms of the other. This is shown in the next example.

Example 3
Solve the simultaneous equations

$3x - 2y = 11$

$y = 5 - 2x$

Solution

Substitute the expression for y given in the second equation, into the first equation.

$$3x - 2(5 - 2x) = 11$$
$$3x - 10 + 4x = 11$$
$$7x = 21$$
$$x = 3$$
$$y = 5 - 2 \times 3$$
$$= 5 - 6$$
$$= -1$$

Multiply out the brackets

Substitute the value for x into the original second equation

The solution is $x = 3$, $y = -1$

One linear and one quadratic equation

When you need to solve a pair of simultaneous equations, one of which is linear and one of which is quadratic, you need to substitute the linear equation into the quadratic equation.

Example 4
Solve the simultaneous equations

$$x^2 + 2y^2 = 6$$
$$x - y = 1$$

Solution

Start by using the linear equation to write one variable in terms of the other.

$$x = y + 1$$

Now substitute this expression for y into the first equation

$$(y+1)^2 + 2y^2 = 6$$
$$y^2 + 2y + 1 + 2y^2 = 6$$
$$3y^2 + 2y - 5 = 0$$
$$(3y+5)(y-1) = 0$$
$$y = -\tfrac{5}{3} \text{ or } y = 1$$

Multiply out, simplify and factorise

Sometimes you will need to use the quadratic formula to solve the resulting quadratic equation.

$y = -\frac{5}{3}$ $\quad x = y + 1 = -\frac{5}{3} + 1 = -\frac{2}{3}$

$y = 1$ $\quad x = y + 1 = 1 + 1 = 2$

Now substitute each value for y into the linear equation to find the corresponding values of x

The solutions are $x = -\frac{2}{3}, y = -\frac{5}{3}$ and $x = 2, y = 1$

The intersection of a line and a curve

Just as the point of intersection of two straight lines can be found by solving the equations of the two lines simultaneously, the point(s) of intersection of a line and a curve can be found by solving their equations simultaneously.

In many cases, the equations of both the line and the curve are given as an expression for y in terms of x. When this is the case, a sensible first step is to equate the expressions for y, as this leads to an equation in x only.

Example 5

Find the coordinates of the points where the line $y = x + 2$ meets the curve $y = x^2\ 3x + 5$.

Solution

$$x^2 - 3x + 5 = x + 2$$
$$x^2 - 4x + 3 = 0$$
$$(x - 3)(x - 1) = 0$$
$$x = 3 \text{ or } x = 1$$

Equate the expressions for y to give an equation in x only

When $x = 3$ then $y = 3 + 2 = 5$
When $x = 1$ then $y = 1 + 2 = 3$.

Substitute the x values into the equation of the line

The points where the line meets the curve are (3, 5) and (1, 3).

You should check that each of these points satisfies the equation of the curve. (You have already used the equation of the line to find the y-values).

Notice that this problem involved solving a quadratic equation, which in this case had two solutions, showing that the line crossed the curve twice. However, the quadratic equation could have had no solutions, which would indicate that the line did not meet the curve at all, or one repeated solution, which would indicate that the line touches the curve.