Section 1: Quadratic graphs and equations

Notes and Examples

These notes contain subsections on

Factorising quadratic expressions

This should be revision of GCSE work. It is essential that you are confident in factorisation.

Example 1

Factorise the expressions

(i) $x^2 + 4x + 3$

(ii) $x^2 - 4x - 12$

(iii) $2x^2 - 7x + 6$

Solution

(i) $x^2 + 4x + 3 = (x\ ......)(x\ .....)$

Start with an x in each bracket

$x^2 + 4x + 3 = (x+1)(x+3)$

You need two numbers whose sum is 4 and whose product is 3. These are +1 and +3.

(ii) $x^2 - 4x - 12 = (x\ ......)(x\ .....)$

Start with an x in each bracket

$x^2 - 4x - 12 = (x+6)(x+2)$

You need two numbers whose sum is –4 and whose product is –12. These are –6 and +2.

(iii) $2x^2 - 7x + 6 = (2x\ .....)(x\ .....)$

In this case you need to start with $2x$ in one bracket and x in the other.

$2x^2 - 7x + 6 = (2x-3)(x-2)$

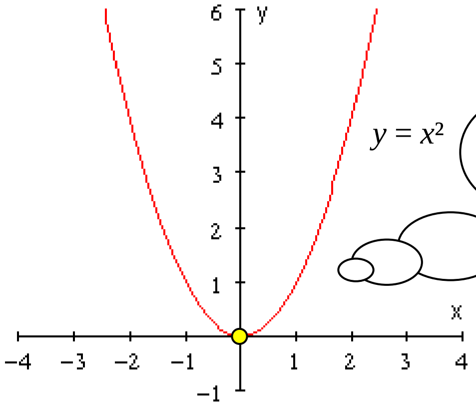
It is not so straightforward to find the two numbers in this case, because of the $2x$ in one bracket. The two numbers must have a product of +6, and as the coefficient of x is negative, they must both be negative. Try the different possibilities (–1 and –6, or –2 and –3, in either order), until you find the correct one.

Try the ***Quadratic factors walkthrough***.

Graphs of quadratic functions

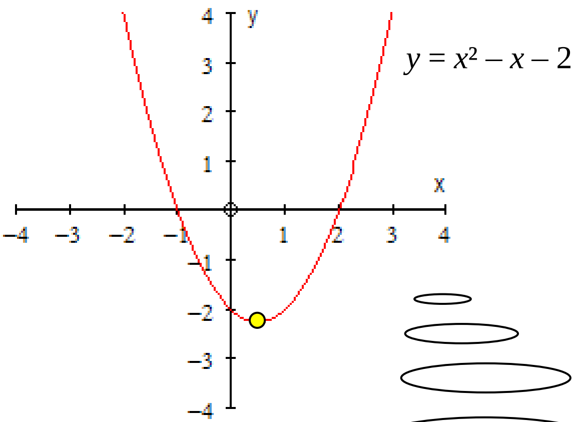
Factorising a quadratic expression gives you information about the graph of a quadratic function. Do not think of this work as just algebraic manipulation, think about it also in terms of the graph of the function. Linking algebra and graphs is a very important mathematical skill; the good news is that being able to consider problems both algebraically and graphically usually makes them easier! A graphical calculator or graphing software such as GeoGebra or Desmos will be very useful.

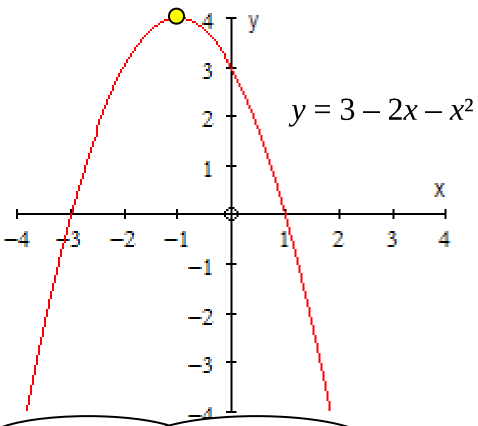
You may already be familiar with the graph of the simplest quadratic function, $y = x^2$.



The curve given by graphs of quadratics is called a ***parabola.*** Notice that all quadratic graphs have reflection symmetry. The mirror line is always a vertical line through the **turning point** or **vertex** of the curve (shown in yellow on these graphs).

All other quadratic graphs have basically the same shape, but they may be "stretched", "squashed", shifted or inverted.





Factorise the equations of these graphs. What is the relationship between the factorised form and the graph? Can you explain this?

Notice that the graphs of functions with a negative x^2 term are inverted (upside down).

Try the ***Quadratic graphs skill pack***.

Solving quadratic equations by factorisation

Solving quadratic equations is important not just from the algebraic point of view, but because it gives you information about the graph of a quadratic function. The roots of the equation $ax^2 + bx + c = 0$ tell you where the graph of the function $y = ax^2 + bx + c$ crosses the x-axis, since these are the points where $y = 0$.

Some quadratic equations can be solved by factorising.

Example 2
Solve these quadratic equations by factorising.
(i) $x^2 + 2x - 8 = 0$ (ii) $2x^2 + 11x + 12 = 0$

Solution
(i)
$$x^2 + 2x - 8 = 0$$
$$(x+4)(x-2) = 0$$
$$x + 4 = 0 \text{ or } x - 2 = 0$$
$$x = -4 \text{ or } x = 2$$

For this expression to be zero, one or other of the factors must be zero.

(ii)
$$2x^2 + 11x + 12 = 0$$
$$(2x+3)(x+4) = 0$$
$$2x + 3 = 0 \text{ or } x + 4 = 0$$
$$x = -\tfrac{3}{2} \text{ or } x = -4$$

Try the ***Quadratic factors walkthrough*** and the ***Solving quadratics by factorisation skill pack***.

Quadratic equations in disguise

Some equations don't immediately look like quadratic equations, but they can be rewritten in quadratic form. Sometimes a substitution can be useful.

Example 3
Solve the equation $x^4 - x^2 - 6 = 0$.

Solution

By putting $x^2 = y$ you can make this into a quadratic equation.

$y^2 - y^2 - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3$ or $y = -2$

$x^2 = 3$ or $x^2 = -2$

$x = \pm\sqrt{3}$

Now change back to using x

Notice that $x^2 = -2$ has no real roots, so the only roots come from $x^2 = 3$

The turning point of a quadratic graph

A quadratic function is usually written in the form $y = ax^2 + bx + c$, where a, b and c are constants. However, writing quadratic functions in different forms can sometimes give you additional information about the function.

You have already seen that writing a quadratic function in factorised form gives you some useful information about the graph of the function. It tells you where the graph crosses the x-axis. This also applies to other polynomial functions.

However, sometimes you may not be interested in where the graph cuts the axes, but you may want to know the coordinates of the maximum or minimum point of the graph (often called the **vertex**). One way to do this is by using the *completed square form* for a quadratic function. This means the form $a(x-p)^2 + q$, so that you have a perfect square involving x, and a constant term.

To help you to visualise how the completed square form relates to the graph of a quadratic function, have a look at the ***Explore: Completing the square*** resource.

Example 4

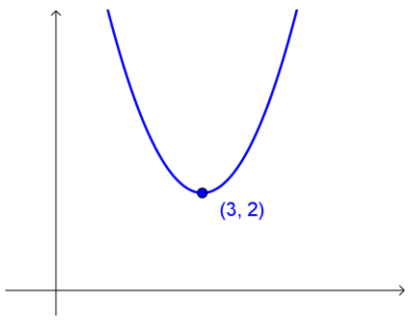
For each of the following quadratic graphs, write down the equation of the line of symmetry of the graph and the coordinates of the vertex (turning point), and hence sketch the graph.

(i) $y = (x-3)^2 + 2$ (ii) $y = (2x-1)^2 - 5$ (iii) $y = 1 - (x+2)^2$

Solution

(i) $y = (x-3)^2 + 2$

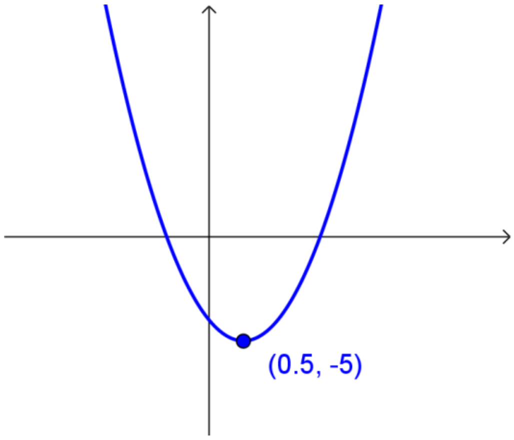
Line of symmetry is $x = 3$.
Minimum point is (3, 2).



(ii) $y = (2x-1)^2 - 5$

Line of symmetry is $x = \frac{1}{2}$.

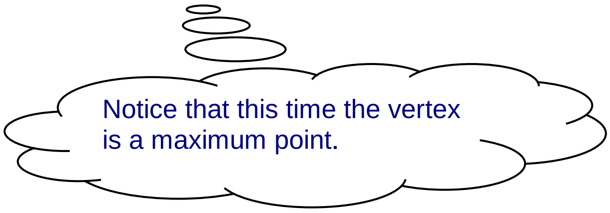
Minimum point is ($\frac{1}{2}$, -5).

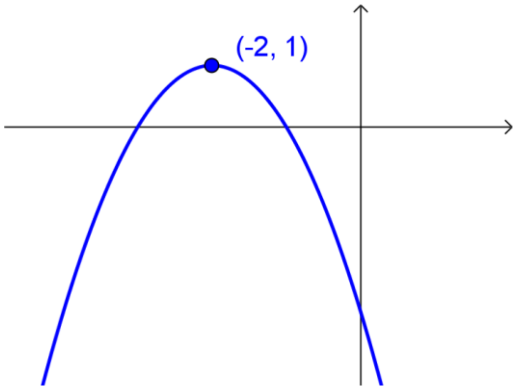


(iii) $y = 1-(x+2)^2$

Line of symmetry is x = -2.

Maximum point is (-2, 1).





Example 5

Find the equations of quadratic graphs with the given turning points. Give the equations in the form $y = ax^2 + bx + c$.

(i) Minimum point (1, -2)

(ii) Minimum point (-3, 1)

(iii) Maximum point (4, 3)

Solution

(i) The equation of the graph is
$$\begin{aligned} y &= (x-1)^2 - 2 \\ &= x^2 - 2x + 1 - 2 \\ &= x^2 - 2x - 1 \end{aligned}$$

(ii) The equation of the graph is
$$\begin{aligned} y &= (x+3)^2 + 1 \\ &= x^2 + 6x + 9 + 1 \\ &= x^2 + 6x + 10 \end{aligned}$$

(iii) The equation of the graph is
$$\begin{aligned} y &= 3-(x-4)^2 \\ &= 3-\left(x^2 - 8x + 16\right) \\ &= 3 - x^2 + 8x - 16 \\ &= -x^2 + 8x - 13 \end{aligned}$$

Completing the square

The examples below show how to write a quadratic function in the completed square form.


Example 6

Write the expression $x^2 + 4x + 7$ in the completed square form.

Solution

First you need to find a quadratic expression which is a perfect square and which begins with $x^2 + 4x$.
You do this by looking at the coefficient of x, in this case 4, and halving it. In this case you get 2.
This tells you that the perfect square you need is $(x + 2)^2$.

$$(x+2)^2 = x^2 + 4x + 4$$
$$x^2 + 4x + 7 = x^2 + 4x + 4 + 3$$
$$= (x+2)^2 + 3$$

$(x+2)^2$ is the 'square'

This is why the technique is called 'completing the square'.

There are several different approaches to writing out the working. They are all basically the same, so if you have learnt a different way which suits you, then stick to it.

The next example shows a situation where the coefficient of x^2 is not 1.

Example 7

Write the expression $2x^2 - 6x + 1$ in the completed square form.

Solution

Start by taking out the coefficient of x^2, in this case 2, as a factor.

$$2x^2 - 6x + 1 = 2[x^2 - 3x] + 1$$

Now look at the expression inside the square bracket. You need to find a quadratic expression which is a perfect square and starts with $x^2 - 3x$. Take the coefficient of x, which is –3, and halve it to get $-\frac{3}{2}$. The perfect square you need is therefore $\left(x - \frac{3}{2}\right)^2$.

$$\left(x-\tfrac{3}{2}\right)^2 = x^2 - 3x + \tfrac{9}{4} \text{ so } x^2 - 3x = \left(x-\tfrac{3}{2}\right)^2 - \tfrac{9}{4}$$

$$\begin{aligned} 2x^2 - 6x + 1 &= 2[(x-\tfrac{3}{2})^2 - \tfrac{9}{4}] + 1 \\ &= 2(x-\tfrac{3}{2})^2 - \tfrac{9}{2} + 1 \\ &= 2(x-\tfrac{3}{2})^2 - \tfrac{7}{2} \end{aligned}$$

In the next example, the coefficient of x^2 is negative. This can be dealt with by taking out a factor -1.

Example 8

Write the expression $5 + x - x^2$ in the completed square form.

Hence sketch the graph of $y = 5 + x - x^2$, showing the coordinates of its turning point.

Solution

Start by taking out –1 as a factor from the first two terms.

$$5 + x - x^2 = -[x^2 - x] + 5$$

Now you need a quadratic expression which is a perfect square and starts with $x^2 - x$. Half the coefficient of x is $-\tfrac{1}{2}$, so the perfect square you need is $(x-\tfrac{1}{2})^2$.

$$(x-\tfrac{1}{2})^2 = x^2 - x + \tfrac{1}{4} \text{ so } x^2 - x = (x-\tfrac{1}{2})^2 - \tfrac{1}{4}$$

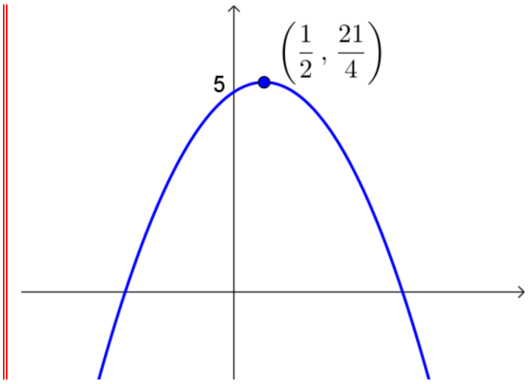
$$\begin{aligned} 5 + x - x^2 &= -[(x-\tfrac{1}{2})^2 - \tfrac{1}{4}] + 5 \\ &= -(x-\tfrac{1}{2})^2 + \tfrac{1}{4} + 5 \\ &= -(x-\tfrac{1}{2})^2 + \tfrac{21}{4} \end{aligned}$$

$y = 5 + x - x^2$ can therefore be written as $y = -(x-\tfrac{1}{2})^2 + \tfrac{21}{4}$.

Since the coefficient of x^2 is negative, the graph has a maximum point rather than a minimum point.

From the completed square form, the graph has maximum point $\left(\tfrac{1}{2}, \tfrac{21}{4}\right)$.

Also note that it passes through the point (0, 5).



Try the ***Completing the square walkthrough*** and the ***Completing the square skill pack***.