Section 2: The quadratic formula

Notes and Examples

These notes contain subsections on

- Solving quadratic equations using the formula
- Problem solving

Solving quadratic equations using the formula

If a quadratic equation is written in the completed square form, it is easy to solve.

Example 1

(i) Write $x^2 - 4x - 5$ in the completed square form.

(ii) Hence solve the equation $x^2 - 4x - 5 = 0$

Solution

(i) $x^2 - 4x - 5 = (x-2)^2 - 4 - 5$

$= (x-2)^2 - 9$

(ii) $(x-2)^2 = 9$

$x - 2 = \pm 3$

$x = 2 \pm 3$

$x = 5$ or -1

However, unless you already have the equation in the completed square form, as in the example above, it is easier to use the quadratic formula, which is just a generalisation of the completing the square method.

The quadratic formula for the solutions of the equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ is called the **discriminant.** This is very important as it tells you something about the nature of the solutions.
In each case the solution(s) correspond to the points where the graph meets the x-axis.

- If the discriminant is positive, then there are two real solutions. (If the discriminant is a positive square number, then the two real solutions are rational and it is possible to solve the equation by factorisation; otherwise the solutions are irrational and you must use the quadratic formula.)

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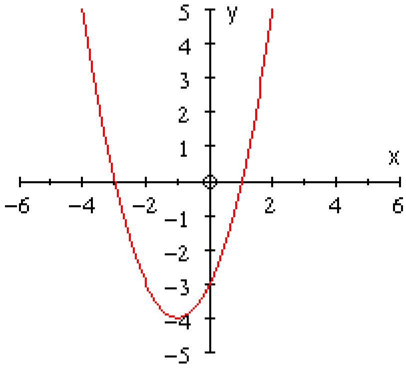
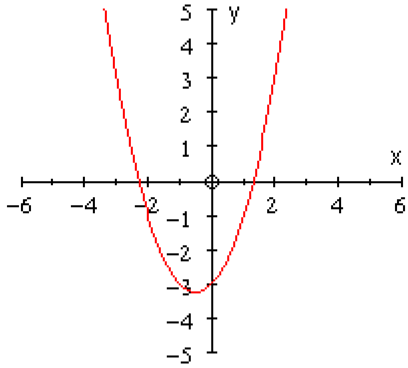
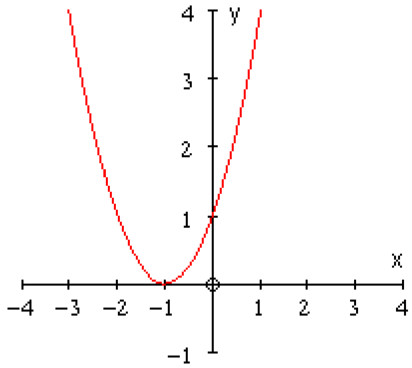


$y = x^2 + 2x - 3$
Discriminant = 16
Two rational solutions



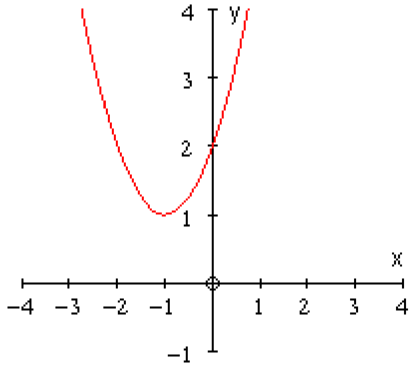
$y = x^2 + x - 3$
Discriminant = 13
Two real, irrational solutions

- If the discriminant is zero, then the quadratic is a perfect square and there is one real solution, which can be found by factorisation.



$y = x^2 + 2x + 1$
Discriminant = 0
One real solution

- If the discriminant is negative, then there are no real solutions.



$y = x^2 + 2x + 2$
Discriminant = -4
No real solutions

As the graph does not meet the x-axis, there cannot be any real solutions.

When you need to solve a quadratic equation, it is useful to quickly work out the discriminant before you start, so that you know whether there are real solutions, and whether the equation can be solved by factorisation.

Your calculator may be able to solve quadratic equations, and some calculators will give the answers in exact form (using surds). However, you may be required to show working in some questions, so you must know the quadratic formula and be confident in using it.

Example 2

For each of the following quadratic equations, find the discriminant and solve the equation, where possible, by a suitable method

(i) $2x^2 - 4x + 1 = 0$ (ii) $6x^2 + 11x - 10 = 0$

(iii) $3x^2 - 2x + 4 = 0$ (iv) $4x^2 + 12x + 9 = 0$

Solution

(i) $a = 2$, $b = -4$, $c = 1$

Discriminant = $(-4)^2 - 4 \times 2 \times 1 = 16 - 8 = 8$

Since the discriminant is positive, there are two real solutions. As it is not a square number, the equation must be solved using the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{8}}{2 \times 2} \\ &= \frac{4 \pm 2\sqrt{2}}{4} \\ &= \frac{2 \pm \sqrt{2}}{2} \end{aligned}$$

(ii) $a = 6$, $b = 11$, $c = -10$

Discriminant = $11^2 - 4 \times 6 \times -10 = 121 + 240 = 361$

Since the discriminant is positive, there are two real solutions. As it is a square number (19^2), the equation can be solved by factorisation.

$$\begin{aligned} &6x^2 + 11x - 10 = 0 \\ &(3x - 2)(2x + 5) = 0 \\ &x = \tfrac{2}{3} \text{ or } x = -\tfrac{5}{2} \end{aligned}$$

(iii) $a = 3$, $b = -2$, $c = 4$

Discriminant = $(-2)^2 - 4 \times 3 \times 4 = 4 - 48 = -44$

Since the discriminant is negative, there are no real solutions.

(iv) $a = 4$, $b = 12$, $c = 9$

Discriminant = $12^2 - 4 \times 4 \times 9 = 144 - 144 = 0$

Since the discriminant is zero, there is one solution and the equation can be solved by factorisation into a perfect square.

$$\begin{aligned} &4x^2 + 12x + 9 = 0 \\ &(2x + 3)^2 = 0 \\ &x = -\tfrac{3}{2} \end{aligned}$$

Try the ***Quadratic formula walkthrough***, and the ***Quadratic formula skill pack***.

Problem solving

Some problems, when translated into algebra, involve quadratic equations.

Example 3
A rectangular box has width 2 cm greater than its length, and height 3 cm less than its length. The total surface area of the box is 548 cm².
What are the dimensions of the box?

Solution
Let the length of the box be x cm.
The width of the box is $x + 2$ cm, and the height is $x - 3$ cm.

The surface area of the box is given by $2x(x+2)+2x(x-3)+2(x+2)(x-3)$.

$$2x(x+2)+2x(x-3)+2(x+2)(x-3)=548$$

Divide through by 2

$$x(x+2)+x(x-3)+(x+2)(x-3)=274$$
$$x^2+2x+x^2-3x+x^2-x-6=274$$
$$3x^2-2x-280=0$$

The discriminant is 3364, which is 58², so this must factorise

$$(3x+28)(x-10)=0$$
$$x=10$$

$3x + 28 = 0$ gives a negative value of x, which does not make sense in this context. So the solution must be $x - 10 = 0$.

The length of the box is 10 cm, the width is 12 cm and the height is 7 cm.

Notice that in Example 3, you could discard one of the possible solutions as a negative solution did not make sense in the context. This is not always the case. In some situations, a negative solution can have a practical meaning. For example if the height of a stone thrown from the edge of a cliff is negative, this simply means that the stone is below the level of the cliff at that point. However, if the stone was thrown from level ground, then a negative height does not make sense.

Some problems leading to quadratic equations do have two possible solutions. Always consider whether your solution(s) make sense in the context.