Section 1: Polynomial functions and graphs

Notes and Examples

These notes contain subsections on

Adding and subtracting polynomials

Example 1

For the polynomials $\mathrm{f}(x) = 2x^3 - 3x^2 + 1$

$\mathrm{g}(x) = x^3 + x^2 - 3x - 4$

find (i) $\mathrm{f}(x) + \mathrm{g}(x)$

(ii) $\mathrm{f}(x) - \mathrm{g}(x)$

Solution

(i) $$\begin{aligned} \mathrm{f}(x) + \mathrm{g}(x) &= (2x^3 - 3x^2 + 1) + (x^3 + x^2 - 3x - 4) \\ &= (2x^3 + x^3) + (-3x^2 + x^2) + (-3x) + (1 - 4) \\ &= 3x^3 - 2x^2 - 3x - 3 \end{aligned}$$

Add the terms in x^3, the terms in x^2, the terms in x and the constant terms separately. Notice that $\mathrm{f}(x)$ does not have a term in x. It is not essential to put in the brackets as shown here, but it can be helpful.

$$\begin{array}{rrrrr} & 2x^3 & -3x^2 & & +1 \\ + & x^3 & +x^2 & -3x & -4 \\ \hline & 3x^3 & -2x^2 & -3x & -3 \end{array}$$

Alternatively you can write it out like an addition sum:

(ii) $$\begin{aligned} \mathrm{f}(x) - \mathrm{g}(x) &= (2x^3 - 3x^2 + 1) - (x^3 + x^2 - 3x - 4) \\ &= (2x^3 - x^3) + (-3x^2 - x^2) + (-3x) + (1 + 4) \\ &= x^3 - 4x^2 - 3x + 5 \end{aligned}$$

Do this in a similar way, but be careful about signs.

$$\begin{array}{rrrrr} & 2x^3 & -3x^2 & & +1 \\ - & x^3 & +x^2 & -3x & -4 \\ \hline & x^3 & -4x^2 & +3x & +5 \end{array}$$

Alternatively (again be careful about signs):

Multiplying polynomials

You are already familiar with multiplying out two linear expressions to obtain a quadratic expression.

You now need to be able to deal with more complicated multiplications. The principle is the same – for a multiplication involving two brackets, each term in one bracket needs to be multiplied by each term in the other bracket.

First of all, it is helpful to think about how many terms there will be in the result (before simplifying). Each of the 3 terms in the first bracket must be multiplied by each of the terms in the second bracket, so there should be 9 terms altogether.

There are a number of different ways of setting out this multiplication. One way is shown in Example 2 below.

Example 2
Multiply x^2+3x-2 by $2x^2-x+4$

Each term in the first bracket is multiplied by the whole of the second bracket

Solution

$$\begin{aligned}(x^2+3x-2)(2x^2-x+4) &= x^2(2x^2-x+4)+3x(2x^2-x+4)-2(2x^2-x+4)\\ &= (2x^4-x^3+4x^2)+(6x^3-3x^2+12x)+(-4x^2+2x-8)\\ &= 2x^4+(-x^3+6x^3)+(4x^2-3x^2-4x^2)+(12x+2x)-8\\ &= 2x^4+5x^3-3x^2+14x-8\end{aligned}$$

9 terms as expected

Collect like terms together

You also need to be able to multiply out expressions involving more than one bracket. This is shown in the next example.

Example 3
Multiply out $(x-2)(2x+3)(3x-1)$

Solution
It is often easiest to multiply out one pair of brackets, and then multiply the result by the third bracket.

$$\begin{aligned}(x-2)(2x+3) &= 2x^2+3x-4x-6\\ &= 2x^2-x-6\end{aligned}$$

Simplify here or the next step will be more complicated!

$$\begin{aligned}&(x-2)(2x+3)(3x-1) = (2x^2-x-6)(3x-1)\\ &= 2x^2(3x-1)-x(3x-1)-6(3x-1)\\ &= 6x^3-2x^2-3x^2+x-18x+6\\ &= 6x^3-5x^2-17x+6\end{aligned}$$

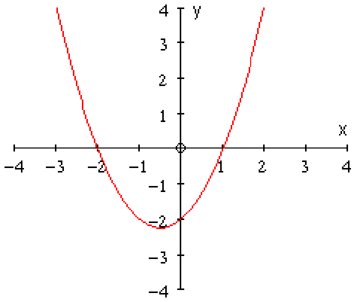
You can of course use another approach, such as using a table, or doing it in your head, to multiply the quadratic by the third bracket.

Graphs of polynomial functions

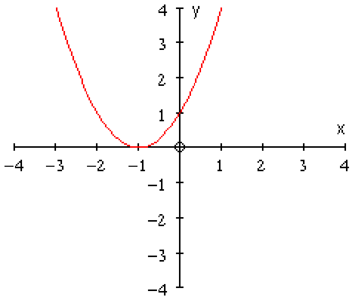
There are two simple rules about the graphs of polynomial functions:

- a polynomial of order n meets the x-axis at most n times.
- a polynomial of order n has at most $n - 1$ turning points.

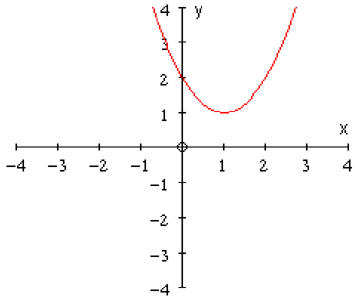
For example, a quadratic, which has order 2, has 1 turning point and meets the x-axis at most twice.



This quadratic crosses the x-axis twice.

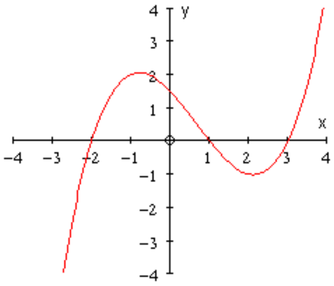


This quadratic touches the x-axis once.

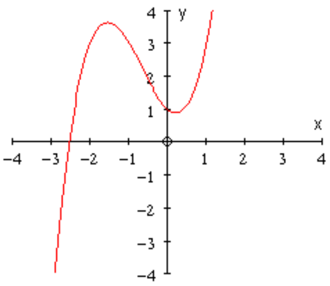


This quadratic does not meet the x-axis at all.

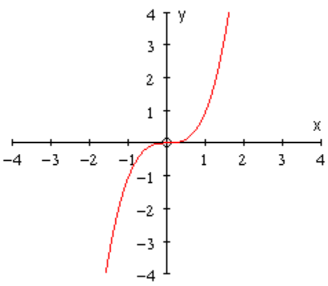
A cubic, which has order 3, has at most 2 turning points and meets the x-axis at most three times and at least once. Here are some examples.



This cubic graph has two turning points and crosses the x-axis three times.



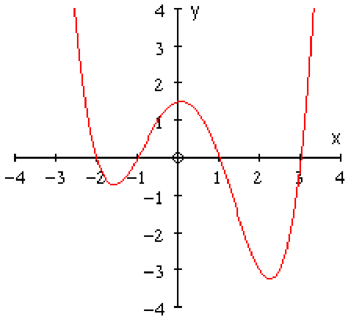
This cubic graph has two turning points, but only crosses the x-axis once.

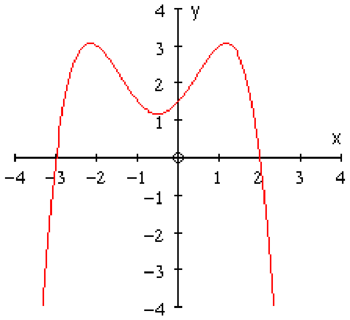


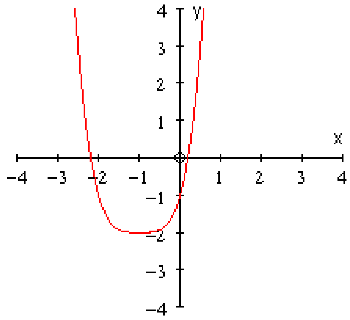
The simplest cubic graph of all, $y = x^3$, has just one turning point, a point of inflection. You can think of this as two turning points in the same place. It crosses the x-axis just once.

Edexcel AS Maths Polynomials 1 Notes and Examples

A quartic, which has order 4, has at most 3 turning points and meets the x-axis at most four times. Here are some examples.







This quartic graph has three turning points and crosses the x-axis four times.

This quartic graph has three turning points and crosses the x-axis twice.

This quartic graph has just one turning point and crosses the x-axis twice. Notice that the turning point is much flatter than usual – it is actually three turning points all in the same place. This makes the shape different from a quadratic graph.

Sketching graphs of polynomials in factorised form

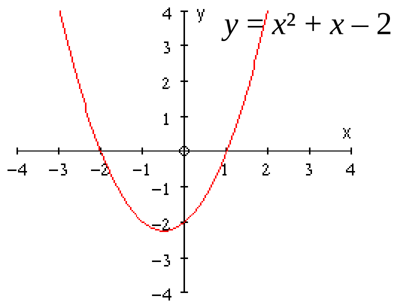
You have already done some work on factorising quadratic expressions. As you know, a quadratic expression can sometimes be factorised into two linear factors. These factors can be used to tell you where the graph of the quadratic cuts the x-axis.

For example, the quadratic expression $\quad y = x^2 + x - 2$
can be written in factorised form as $\quad y = (x-1)(x+2)$

When the graph cuts the x-axis, the value of $y = 0$. $\quad (x-1)(x+2) = 0$
$x = 1 \text{ or } x = -2$
The graph therefore cuts the x-axis at (1, 0) and (-2, 0).

You can also find out where the graph cuts the y-axis by substituting $x = 0$. In this case, when $x = 0$, $y = -2$, so the graph cuts the y-axis at (0, -2).

This information allows you to sketch the graph.



It is also useful to think about the behaviour of the graph as $x \to \infty$ (i.e. when x is very large and positive) and as $x \to -\infty$ (i.e. when x is very large and negative). For the quadratic graph above, for values of x which are numerically large the term x^2 is the most significant term (the dominant term). So you just need to think about whether this term is positive or negative. In this case it is positive as $x \to \infty$ and as $x \to -\infty$. So the graph disappears off the top of the page at both the left and right of the graph.

The same ideas can be extended to any polynomial. You will learn how to factorise cubics and higher order polynomials in Section 2. For now, you will look at polynomials that are given in factorised form.

Examination questions commonly ask you to sketch the graph of a function which you have already factorised.

This should be an easy source of marks (provided you have managed the factorising!) but a lot of candidates throw marks away. Here are some tips:

- A sketch means a sketch! Do it in the answer booklet, not on graph paper (if graph paper is provided that
- does not mean that you are expected to use it!). You should certainly not be calculating and plotting points – this is very time-consuming, unnecessary and may result in only part of the graph.

- Show the points at which the graph cuts BOTH axes. The points at which the graph cuts the x-axis can be found from the factorised equation. Find the point at which the graph cuts the y-axis by substituting $x = 0$, and mark on this point – one mark is often given for this.

- Make sure that your graph is the right way up. If you have marked on the point where the graph cuts the y-axis then it should be clear how the graph goes. Also think about whether y is positive or negative when x is very large and positive, and when x is very large and negative.

- Do not stop the graph when you reach an axis! You may lose marks for this. The graph should go beyond each of the points marked.

- If you have a graphical calculator, do use it to check, but don't rely on it. Sometimes the output can be misleading - the scale used may mean that not all the important features are visible. If the question requires you to calculate exact values of intersections, then you will need to show your working and not just read off the graph.

Example 4

Sketch the following graphs, showing the points where the graphs meet the coordinate axes.

(i) $y=(x-1)(x+3)(2x+1)$

(ii) $y=x(x-2)^2(x+2)$

Solution

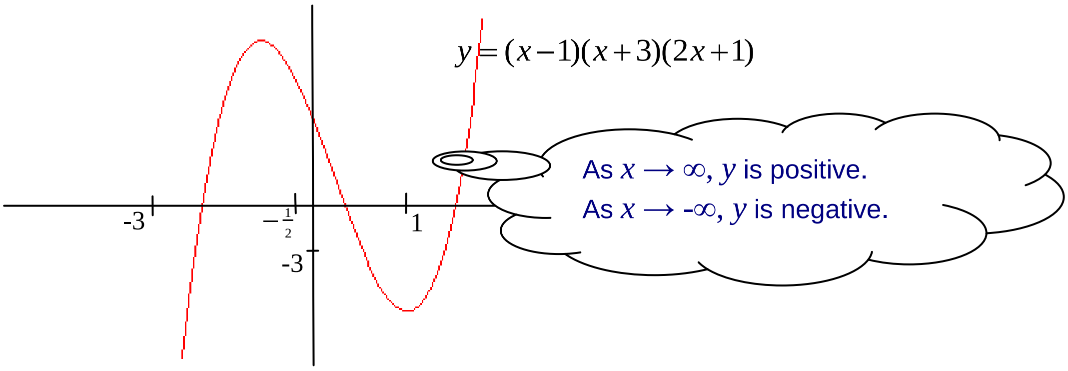
(i) $y=(x-1)(x+3)(2x+1)$

When y = 0, x = 1, x = -3 or $x = -\frac{1}{2}$.

When x = 0, y = -1 × 3 × 1 = -3

The graph crosses the x-axis at (1, 0), (-3, 0) and $(-\frac{1}{2}, 0)$.

The graph crosses the y-axis at (0, -3).



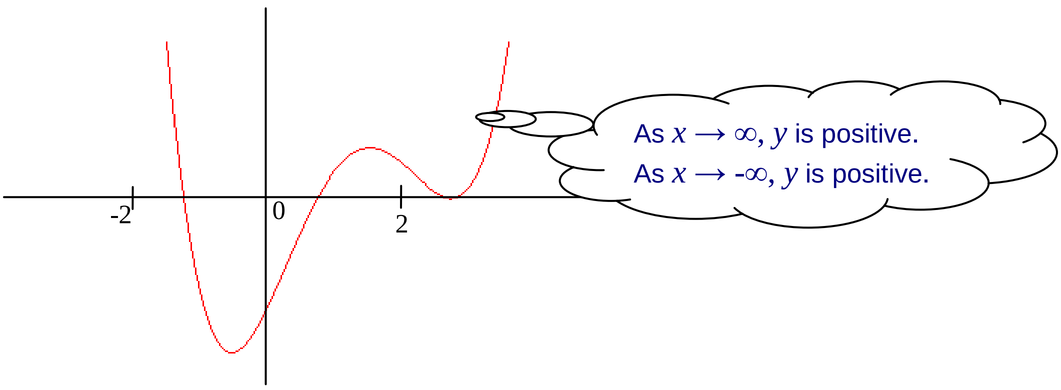
(ii) $y=x(x-2)^2(x+2)$

When y = 0, x = 0, x = 2 (repeated) or x = -2.

When a root is repeated, the graph touches the x-axis

There is no need to look at where the graph crosses the y-axis, since you already know it passes through the origin.

The graph crosses the x-axis at (0, 0) and (-2, 0), and touches the x-axis at (2, 0).



Finding the equation of a curve

If you are know where a polynomial curve cuts the axes (or alternatively, if you are given the roots of a polynomial equation), you can deduce the equation of the curve by writing it in factorised form and then multiplying out the brackets.

Example 5
Find a polynomial equation which has roots $x = 1$, $x = -3$ and $x = 0.5$.

Solution
A polynomial equation with these roots is $(x-1)(x+3)(2x-1)=0$.

Multiplying out:
$$\begin{aligned}(x-1)(x+3)(2x-1) &= (x^2+2x-3)(2x-1)\\ &= 2x^3+4x^2-6x-x^2-2x+3\\ &= 2x^3+3x^2-8x+3\end{aligned}$$

An equation with these roots is $2x^3+3x^2-8x+3=0$

Note that any multiple of this polynomial (e.g. $4x^3+6x^2-16x+6=0$) would have the same roots. This is the simplest possible equation.
.