Section 1: Polynomial functions and graphs

Exercise level 3 (Extension)

1. John is investigating a degree 5 polynomial, and its associated graph

$$y = \tfrac{1}{120}x^5 - \tfrac{1}{6}x^3 + x$$

(i) Sketch a degree 5 polynomial graph with positive coefficient of x^5 and with the greatest possible number of local maxima and minima.

(ii) John's first enquiry was to substitute (-*x*) for *x*. Deduce what John found out about the shape of the graph.

(iii) John's second step was to investigate the intercepts of the graph with the *x*-axis. Factorise the polynomial, and prove that there is only one real solution for the equation

$$\tfrac{1}{120}x^5 - \tfrac{1}{6}x^3 + x = 0$$

(iv) Next, John used a spreadsheet to further investigate the shape of the graph. He produced the following table:

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
y	0.00	0.48	0.84	1.00	0.93	0.71	0.53	0.73	1.87	4.69

He used the table to make very rough estimates of the positions of any local minima and maxima. Write down approximations for the coordinates.

(v) From these coordinates, and his deductions in (ii) and (iii), John made a new sketch of the graph. Sketch the graph using your deductions.

(vi) Finally, John was excited to spot that the *x*-values of his intercept and maxima and minima seemed to be close to some of those of $y = \sin x$ where *x* is given in radians. Use a calculator (in 'radians' mode) to compare the values of the graph with the values of $y = \sin x$ when $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$, giving your answers to 3 decimal places. Sketch the two graphs on the same set of axes, and suggest for what values of *x* the polynomial would be a reasonable approximation for sin *x*.

2. Two researchers are separately carrying out the same experiment, and are each trying to find a polynomial graph to represent their results.

(i) Jane finds that her experiments yield the three data points A (2, 2), B (3, 2) and C (6, -2). She finds that all three data points fit on a quadratic graph which passes through the origin with equation

$$y = ax^2 + bx$$

Find the equation of her graph, and draw a sketch of it.

(ii) Samira also discovers the three data points A, B and C, but she has time to find one additional data point D (0 , 3). With the extra point, Samira is able

to find the following cubic polynomial which is satisfied by all four data points

$$y = -\tfrac{1}{12}x^3 + \tfrac{7}{12}x^2 - \tfrac{4}{3}x + 3$$

Draw a sketch of Samira's graph.

(iii) In checking Jane's and Samira's results, Mary decides to 'subtract' Jane's polynomial from that of Samira, to obtain another cubic polynomial and graph. Find Mary's new polynomial and write down its roots. Sketch the graph of Mary's new polynomial.

3. **(For this question you should use a graphical calculator or graphing software.)**
Research into the history of a city suggests that its population over the last few centuries is given approximately in the following table:

Date	1700	1800	1900	2000
x	0	1	2	3
Population (millions) ***y***	1.0	1.5	2.5	4.5

These data could be modelled by using either a quadratic or a cubic polynomial graph, and my computer gives me approximations with equations

$$y = \tfrac{3}{8}x^2 + \tfrac{1}{40}x + \tfrac{41}{40}$$

$$y = \tfrac{5}{60}x^3 + \tfrac{5}{12}x + 1$$

(i) Use your calculator or computer to draw the original points (x , y) and to add the two approximations.

(ii) What does each approximate polynomial suggest was the population of the city in 1600 and 1500? Comment on the results.

(iii) In fact, the data are a very close fit to a non-polynomial graph

$$y = a(b + 2^x)$$

Find the values of a and b.

(iv) What does each of the three separate models predict as the population of the city in 2100?