Edexcel AS Mathematics Polynomials

Section 2: Dividing and factorising polynomials

Notes and Examples

These notes contain subsections on

- Dividing polynomials
- The factor theorem

Dividing polynomials

When you divide one polynomial by another, it may divide exactly, or there may be a remainder, just as in arithmetic.

For example: 26 ÷ 6 = 4 remainder 2

> 26 is called the **dividend**,
> 6 is called the **divisor**,
> 4 is the **quotient**
> and 2 is the **remainder**.

You could rewrite this statement as:

26 = 6 × 4 + 2

The examples you will meet will usually involve dividing a polynomial by a linear expression which is a factor of the polynomial, so there is no remainder.

When you divide a polynomial by a linear expression, the quotient is of order one less than the dividend (e.g. for a quartic, the quotient is cubic)

So $$\frac{\text{cubic polynomial}}{\text{linear expression}} = \text{quadratic expression}$$

which means that $\text{cubic polynomial} = \text{linear expression} \times \text{quadratic expression}$

There are several methods that you can use to divide one polynomial by another. Two are shown in the example below.

Example 1
Divide $2x^3 + 3x^2 + x + 6$ by $x + 2$

> When you divide a cubic expression by a linear expression, as in this example, the quotient is a quadratic expression.

Solution
Method 1: inspection

$2x^3 + 3x^2 + x + 6 = (x+2) \times \text{quadratic expression}$

$2x^3 + 3x^2 + x + 6 = (x+2)(2x^2 + \ldots\ldots + \ldots\ldots)$

> The first term of the quadratic must be $2x^2$ to give the $2x^3$ term in the cubic

$2x^3 + 3x^2 + x + 6 = (x+2)(2x^2 + \ldots\ldots + 3)$

> The last term of the quadratic must be 3 to give the 6 in the cubic

$2x^3 + 3x^2 + x + 6 = (x+2)(2x^2 - x + 3)$

> $2 \times 2x^2 = 4x^2$, so the middle term must be $-x$ to give the right total of $3x^2$ in the cubic. Check the term in x as well!

So the quotient is $2x^2 - x + 3$.

Method 2: long division

$$
\begin{array}{r}
2x^2 - x + 3 \\
x+2\,\overline{)\,2x^3 + 3x^2 + x + 6} \\
\underline{2x^3 + 4x^2} \qquad\quad \\
-x^2 + x \quad\;\; \\
\underline{-x^2 - 2x} \quad\;\; \\
3x + 6 \\
\underline{3x + 6}
\end{array}
$$

This is set out in a similar way to numerical long division

Divide $2x^3$ by x (the first term of the linear expression) to give $2x^2$. Then multiply $x + 2$ by $2x^2$ and subtract.

Divide $-x^2$ by x to give $-x$. Then multiply $x + 2$ by $-x$ and subtract.

As expected there is no remainder

It can be quite difficult to understand what is going on in algebraic long division. The key is to realise that division is equivalent to repeated subtraction – for example, if you want to divide 2688 by 21, you could do it by repeatedly subtracting 21 and keeping a count of how many times you have done it. This would of course take a long time. So long division speeds up this process, by subtracting multiples of 21.

$$
\begin{array}{r}
128 \\
21\,\overline{)\,2688} \\
\underline{2100} \\
588 \\
\underline{420} \\
168 \\
\underline{168}
\end{array}
$$

Here you are subtracting 100 lots of 21. Often when you write out long division you don't put the zeros in

Here you are subtracting 20 lots of 21.

Here you are subtracting 8 lots of 21. So you have subtracted 128 lots of 21 in total

In algebraic division, you are doing much the same thing. In the example above, you are repeatedly subtracting $x+2$. In the first step, you are subtracting $2x^2$ lots of $x+2$ from the original cubic, i.e. you are subtracting $2x^3+4x$ from $2x^3+3x^2+x+6$. This eliminates the term in x^3. After you have done the subtraction you are left with $-x^2+x+6$ (the $+6$ isn't usually written in the division, just as the zeros aren't usually written in the numerical long division). So you then subtract $-x$ lots of $x+2$ from $-x^2+x+6$, and so on.

The factor theorem

If $(x-a)$ is a factor of f(x), then f(a) = 0 and $x = a$ is a root of the equation f(x) = 0.

Conversely, if f(a) = 0, then $(x-a)$ is a factor of f(x).

The factor theorem probably won't come as any great surprise to you. After all, you already know that you can solve some quadratics by factorising them.

e.g. to solve the quadratic equation $x^2 + 3x - 10 = 0$

you factorise: $(x+5)(x-2) = 0$

and deduce the solutions $x = -5$ and $x = 2$

Clearly, for $\mathrm{f}(x) = x^2 + 3x - 10$, $\mathrm{f}(-5) = 0$ and $\mathrm{f}(2) = 0$.

$(x+5)$ is a factor of $\mathrm{f}(x) \Leftrightarrow \mathrm{f}(-5) = 0$

$(x-2)$ is a factor of $\mathrm{f}(x) \Leftrightarrow \mathrm{f}(2) = 0$

The factor theorem simply extends this idea to other polynomials such as cubics, and provides a method for solving cubics and higher polynomials.

Example 2

(i) Solve the equation $x^3 + 2x^2 - 5x - 6 = 0$

(ii) Sketch the graph of $y = x^3 + 2x^2 - 5x - 6$

Solution

(i)

The first step is to find one solution by trial and error.
If there is an integer solution $x = a$, then by the factor theorem $(x - a)$ must be a factor of $x^3 + 2x^2 - 5x - 6$. So a must be a factor of 6. a could therefore be 1, -1, 2, -2, 3, -3, 6 or –6.

Let $\mathrm{f}(x) = x^3 + 2x^2 - 5x - 6$

You need to find a value of x for which $\mathrm{f}(x) = 0$.

$$\mathrm{f}(1) = 1^3 + 2\times1^2 - 5\times1 - 6$$
$$= 1 + 2 - 5 - 6 \neq 0$$
$$\mathrm{f}(-1) = (-1)^3 + 2\times(-1)^2 - 5\times(-1) - 6$$
$$= -1 + 2 + 5 - 6 = 0$$

$\mathrm{f}(-1) = 0$ so by the factor theorem $x + 1$ is a factor of $\mathrm{f}(x)$

The next step is to factorise $\mathrm{f}(x)$ into the linear factor $x + 1$ and a quadratic factor.

$$x^3 + 2x^2 - 5x - 6 = (x+1)\times\text{quadratic factor}$$

$$x^3 + 2x^2 - 5x - 6 = (x+1)(x^2 + x - 6)$$

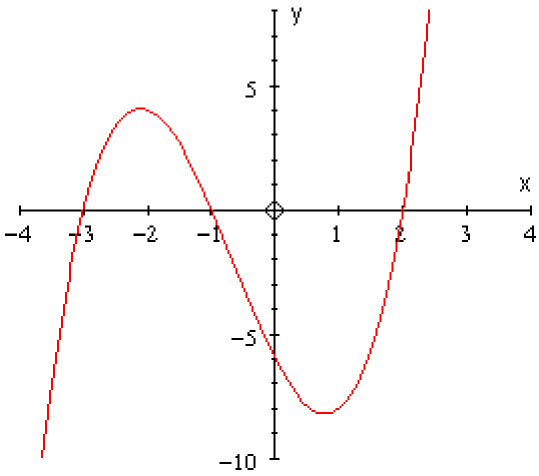
You can do this by inspection or by long division, as in Example 1

$$x^3 + 2x^2 - 5x - 6 = 0$$
$$(x+1)(x^2 + x - 6) = 0$$
$$(x+1)(x+3)(x-2) = 0$$

Factorise the quadratic factor

The roots of the equation are $x = -1, x = -3$ and $x = 2$.

(ii) Part (i) shows that the graph of $y = x^3 + 2x^2 - 5x - 6$ crosses the x-axis at (-3, 0), (-1, 0) and (2, 0). By putting $x = 0$ you can see that it crosses the y-axis at (0, -6). This information allows you to sketch the graph.



Example 3

$f(x) = 2x^3 + px^2 + 5x - 6$ has a factor $x - 2$.

Find the value of p and hence factorise f(x) as far as possible.

Solution

$x - 2$ is a factor of f(x) $\Leftrightarrow$ f(2) = 0

$f(2) = 16 + 4p + 10 - 6$

$\quad = 20 + 4p$

$20 + 4p = 0 \Rightarrow p = -5$

$f(x) = 2x^3 - 5x^2 + 5x - 6$

$\quad = (x - 2)(2x^2 - x + 3)$

The discriminant of the quadratic factor is $(-1)^2 - 4 \times 2 \times 3 = 1 - 24 = -23$

As this is negative, the quadratic factor cannot be factorised further.