Section 2: Notation and proof

Notes and Examples

These notes contain subsections on

Mathematical language

In this section you are introduced to the symbols $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$. You have probably seen these used before – they are used in most mathematics textbooks at this level.

You will probably find using the symbols $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$ fairly straightforward. The use of the words "necessary" and "sufficient" may be a little harder to understand.

Here are a few examples which may help you. You may need to read them through more than once!

1. *A number ends in 5* $\Rightarrow$ *the number is divisible by 5.*

"A number ends in 5" is a sufficient condition for "the number is divisible by 5", since there are no numbers which end in 5 which are not divisible by 5. However, it is not a necessary condition, there are numbers which do not end in 5 which are divisible by 5 (numbers which end in zero).

You can express this the other way round:

A number is divisible by 5 $\Leftarrow$ *the number ends in 5.*

"A number is divisible by 5" is a necessary condition for "the number ends in 5", since all numbers which end in 5 are divisible by 5. However, it is not a sufficient condition, as not all numbers divisible by 5 end in 5.

2. *A number is even* $\Leftarrow$ *the number is divisible by 4.*

"A number is even" is a necessary condition for "the number is divisible by 4", since all numbers which are divisible by 4 are even. However, it is not a sufficient condition, as not all even numbers are divisible by 4.

Again, you can write this the other way round:

A number is divisible by 4 $\Rightarrow$ *the number is even.*

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“A number is divisible by 4” is a sufficient condition for “the number is even”, since there are no numbers which are divisible by 4 which are not even. However, it is not a necessary condition, as there are even numbers which are not divisible by 4.

3. *A number is divisible by 10 $\Leftrightarrow$ the number ends in a zero.*

“A number is divisible by 10” is a necessary and sufficient condition for “the number ends in zero”. All numbers which are divisible by 10 end in zero, and all numbers which end in zero are divisible by 10.
Another way of expressing this is the statement “A number is divisible by 10 if and only if the number ends in zero.”

In the table below, all the statements shown in the same column are equivalent to each other.

A is a necessary condition for B	A is a sufficient condition for B	A is a necessary and sufficient condition for B
If B is true, then A must also be true	If A is true, then B must also be true	A is true if and only if B is true.
A $\Leftarrow$ B	A $\Rightarrow$ B	A $\Leftrightarrow$ B
A is implied by B or A follows from B	A implies B	A implies and is implied by B

You can often use the symbols $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$ when you are writing out a solution to a mathematical problem. For example, if you want to multiply out the equation $y=(2x+3)(x-1)$, you might write:

$$\begin{aligned} & y=(2x+3)(x-1) \\ \Rightarrow\quad & y=2x^2+3x-2x-3 \\ \Rightarrow\quad & y=2x^2+x-3 \end{aligned}$$

In fact, you could use $\Leftrightarrow$ instead of $\Rightarrow$.

$$\begin{aligned} & y=(2x+3)(x-1) \\ \Leftrightarrow\quad & y=2x^2+3x-2x-3 \\ \Leftrightarrow\quad & y=2x^2+x-3 \end{aligned}$$

People often use $\Rightarrow$ when they could use $\Leftrightarrow$, if they are only interested in the logical steps in one direction.

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The converse of a theorem

It should be clear from the examples above that just because a theorem, or a statement, is true, does not necessarily mean that the converse is also true.

If a theorem can be written using $\Leftrightarrow$, then both the theorem and its converse are true.

For example:

A triangle has equal sides $\Leftrightarrow$ *the triangle has equal angles.*

Both this statement and its converse are true.

The statement

An equation is linear $\Rightarrow$ *the equation has exactly one real root*

is true, since all linear equations have exactly one real root. However, the converse of this statement would be

An equation has exactly one real root $\Rightarrow$ *the equation is linear*

which is not true, since there are many examples of quadratics, cubics and indeed polynomial equations of any order which have exactly one real root.

This statement cannot be written using $\Leftrightarrow$.

Proof

Proof is a very important aspect of mathematics and you are expected to have an idea about what proof involves.

The most important thing to realise is that checking lots of cases does not prove that the result is true.

As an example, think of three consecutive even numbers (such as 4, 6 and 8 or 22, 24 and 26) and add them up. You should find that this sum is divisible by 6. Suppose you want to prove that the sum of three consecutive even integers is always divisible by 6. You could test quite a lot of sets of numbers yourself, or you could program a computer to test a very large number of sets of numbers. The computer could keep checking numbers up to astronomically large numbers, but you would still not have checked every single number, and you never can, since there are an infinite number of sets of three consecutive integers! At this stage you could feel sure that the conjecture is in fact true, but to prove it you need to show that it is true for all possible sets of numbers.

Fortunately, there is a method for this.

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Example 1
Prove that the sum of any three consecutive even integers is divisible by 6.

Solution
Any even number is twice a whole number so let the first number be $2n$, where n is an integer.
Then the second number is $2n + 2$, and the third number is $2n + 4$.
The sum of the three numbers is $2n + 2n + 2 + 2n + 4 = 6n + 6 = 6(n + 1)$
$6(n + 1)$ is divisible by 6 for all values of n.

This is an example of direct proof, or proof by deduction. In this case the proof consists of a set of logical steps.

Proof by exhaustion, where there are a limited number of possibilities which can all be tested, is another method of proof.

Here is an example

Example 2
Prove that even number from between 4 and 50 is the sum of two prime numbers.
(Note - is conjectured but not proved that ***every*** *even number is the sum of two primes – this is known as Goldbach's conjecture.)*

Solution
This is shown directly for every case here:

4 = 2 + 2	6 = 3 + 3	8 = 5 + 3	10 = 5 + 5	12 = 7 + 5
14 = 7 + 7	16 = 11 + 5	18 = 11 + 7	20 = 13 + 7	22 = 11 + 11
24 = 13 + 11	26 = 13 + 13	28 = 17 + 11	30 = 17 + 13	32 = 29 + 3
34 = 17 + 17	36 = 23 + 13	38 = 19 + 19	40 = 23 + 17	42 = 23 + 19
44 = 41 + 3	46 = 23 + 23	48 = 41 + 7	50 = 43 + 7	

To disprove a conjecture, all you need is to find a single instance where the conjecture is not true. This is called a counter example.

Here is a conjecture which is not true:

> "For any positive integer n, the sum of any n consecutive integers is a multiple of n."

Here is a counter example which disproves it.

With $n = 4$, $1 + 2 + 3 + 4 = 10$ is the sum of 4 consecutive integers but this is not a multiple of 4.