Edexcel AS Mathematics Graphs and transformations

Section 2: Transformations of graphs

Notes and Examples

These notes contain subsections on:

- Vertical translations of the form $y = \mathrm{f}(x) + a$
- Horizontal translations of the form $y = \mathrm{f}(x - a)$
- Combined translations
- Vertical one-way stretches of the form $y = a\mathrm{f}(x)$
- Horizontal one-way stretches of the form $y = \mathrm{f}(ax)$
- Reflections
- Transformations of trigonometric graphs

Vertical translations of the form y = f(x) + a

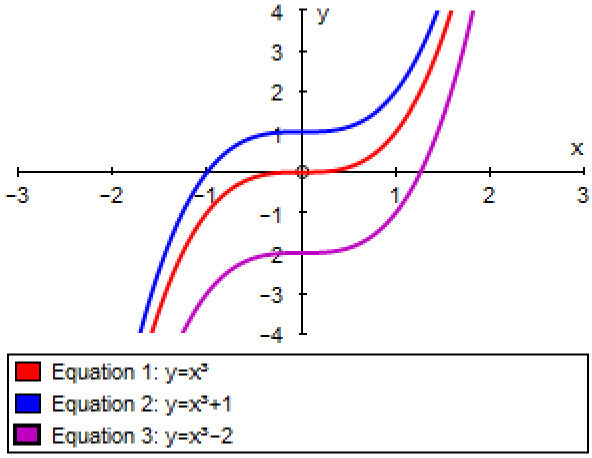
Start by using the ***Explore: Transformations*** resource to investigate graphs of the form $y = \mathrm{f}(x) + a$.

When the curve $y = \mathrm{f}(x)$ is transformed onto the curve $y = \mathrm{f}(x) + a$, for any particular value of x, a is added to the value of y. This has the effect of moving the whole curve a units upwards if a is positive, and a units downwards if a is negative. This is a translation of a units parallel to the y-axis, or, using vector notation, a translation of $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

In general:

> For any function $\mathrm{f}(x)$, the graph of $y = \mathrm{f}(x) + a$ can be obtained from the graph of $y = \mathrm{f}(x)$ by translating it through a units in the positive y direction.

The diagram below shows a graph $y = \mathrm{f}(x)$ in red (in this case $\mathrm{f}(x) = x^3$), the graph $y = \mathrm{f}(x) + 1$ in blue, and the graph $y = \mathrm{f}(x) - 2$ in purple.



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Horizontal translations of the form $y = \mathrm{f}(x - a)$

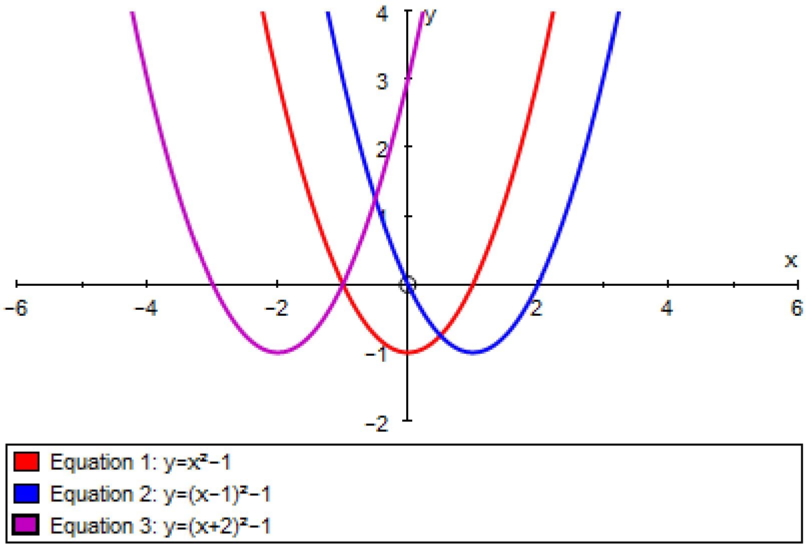
Use the ***Explore: Transformations*** resource to investigate graphs of the form $y = \mathrm{f}(x - a)$.

When the curve $y = \mathrm{f}(x)$ is transformed into the curve $y = \mathrm{f}(x - a)$, for any particular value of y, the value of x must be a units greater to obtain the same value of y. This has the effect of moving the whole curve a units to the right if a is positive, and a units to the left if a is negative. This is a translation of a units parallel to the x-axis, or, using vector notation, a translation of $\begin{pmatrix} a \\ 0 \end{pmatrix}$.

In general:

> For any function $\mathrm{f}(x)$, the graph of $y = \mathrm{f}(x - a)$ can be obtained from the graph of $y = \mathrm{f}(x)$ by translating it through a units in the positive x direction.

The diagram below shows a graph $y = \mathrm{f}(x)$ in red (in this case $\mathrm{f}(x) = x^2 - 1$), the graph $y = \mathrm{f}(x - 1)$ in blue, and the graph $y = \mathrm{f}(x + 2)$ in purple.



Combined translations

Use the ***Explore: Transformations*** resource again, this time varying the values of both p and q to translate the graph in both the x and y directions. What vector describes each transformation?

Translating the graph $y = \mathrm{f}(x)$ by the vector $\begin{pmatrix} s \\ t \end{pmatrix}$ (i.e. s units to the right and t units vertically upwards) gives the graph $y = \mathrm{f}(x - s) + t$. This is simply a combination of the two translations already discussed.
In general:

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For any function f(x), the graph of $y = \mathrm{f}(x - s) + t$ can be obtained from the graph of $y = \mathrm{f}(x)$ by translating it through s units in the positive x direction and t units in the positive y direction.

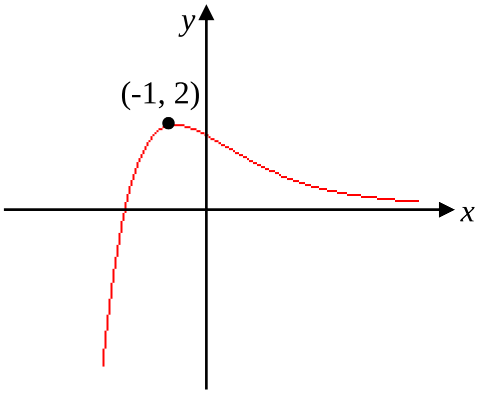
You can rewrite this last result as:

When the graph of $y = \mathrm{f}(x)$ is translated through s units in the positive x direction and t units in the positive y direction, the resulting graph is given by $y - t = \mathrm{f}(x - s)$.

This shows the symmetry of the results. Translating the graph through s units in the positive x direction is equivalent to replacing x with $x - s$. Translating the graph through t units in the positive y direction is equivalent to replacing y with $y - t$.

These ideas can be generalised to any graph.

Example 1
The diagram below shows the graph $y = \mathrm{f}(x)$.



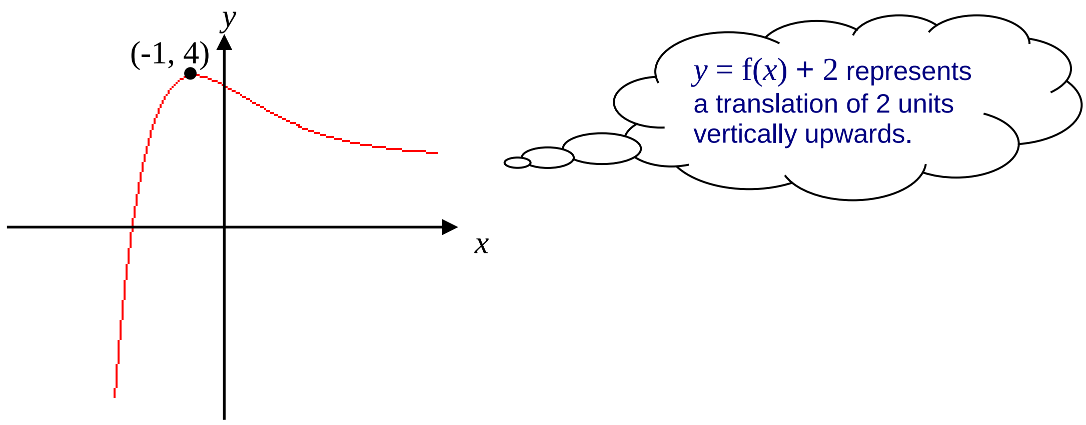
Sketch the graphs of :

(i) $y = \mathrm{f}(x) + 2$
(ii) $y = \mathrm{f}(x) - 1$
(iii) $y = \mathrm{f}(x - 2)$
(iv) $y = \mathrm{f}(x + 1)$
(v) $y = \mathrm{f}(x - 1) - 2$

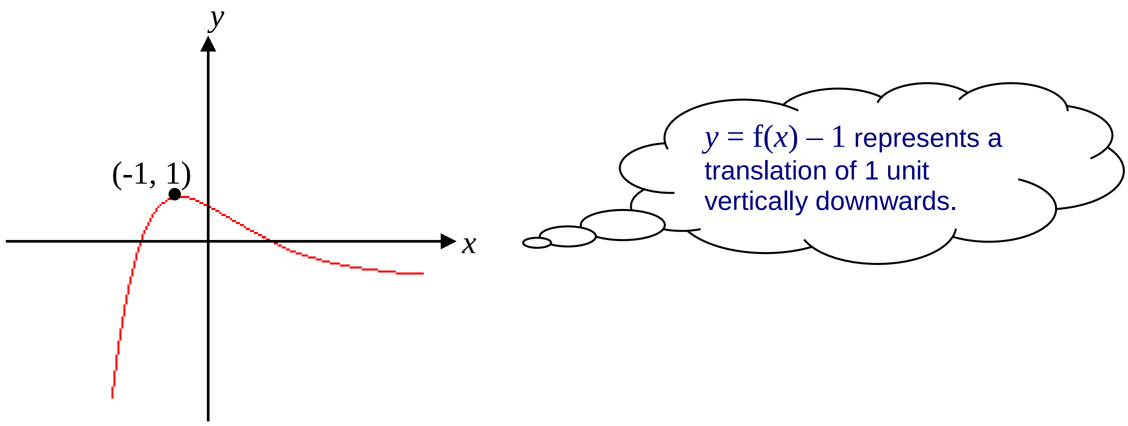
showing the coordinates of the turning point in each case.

Solution

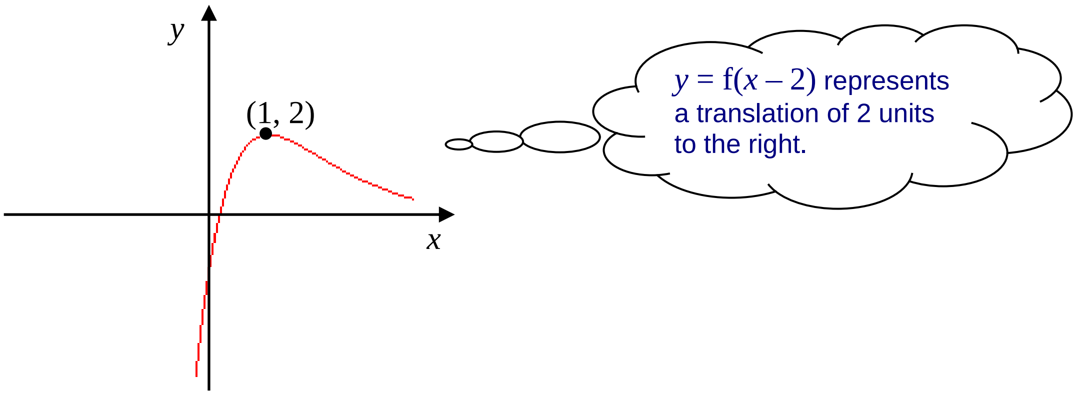
(i)



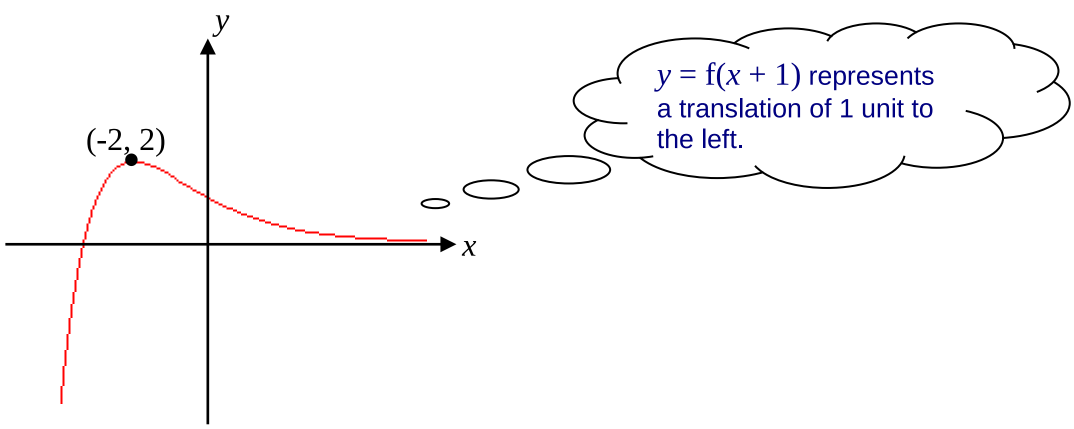
(ii)



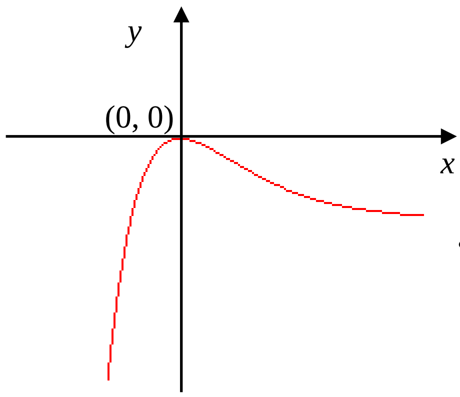
(iii)



(iv)



(v)



$y = \mathrm{f}(x - 1) - 2$ represents a translation of 1 unit to the right and 2 units vertically downwards.

Vertical one-way stretches of the form *y* = *a*f(*x*)

Use the ***Explore: Transformations*** resource to investigate graphs of the form $y = a\mathrm{f}(x)$.

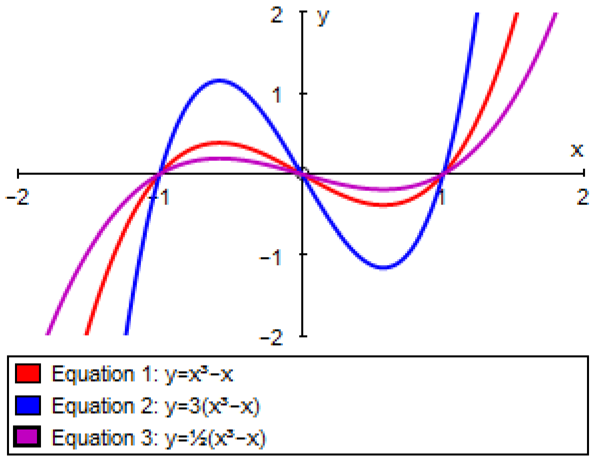
When the curve $y = \mathrm{f}(x)$ is transformed into the curve $y = a\mathrm{f}(x)$, for any particular value of x, the value of y is multiplied by a. This has the effect of stretching the curve by a scale factor of a in the y direction. (Of course, if a is less than 1, then the graph will be compressed rather than stretched).

In general:

> For any function $\mathrm{f}(x)$, and any positive value of a, the graph of $y = a\mathrm{f}(x)$ can be obtained from the graph of $y = \mathrm{f}(x)$ by a stretch of scale factor a parallel to the y-axis.

You can rewrite $y = a\mathrm{f}(x)$ as $\dfrac{y}{a} = \mathrm{f}(x)$. So replacing y with $\dfrac{y}{a}$ results in a stretch of scale factor a parallel to the y-axis.

The diagram below shows a graph $y = \mathrm{f}(x)$ in red (in this case $\mathrm{f}(x) = x^3 - x$), the graph $y = 3\mathrm{f}(x)$ in blue, and the graph $y = \frac{1}{2}\mathrm{f}(x)$ in purple.



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Horizontal one-way stretches of the form *y* = f(*ax*)

Use the ***Explore: Transformations*** resource to investigate graphs of the form $y = \mathrm{f}(ax)$.

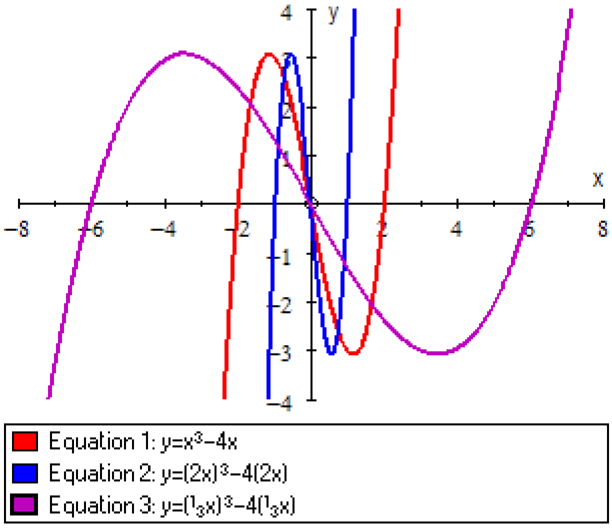
When the curve $y = \mathrm{f}(x)$ is transformed into the curve $y = \mathrm{f}(ax)$, for any particular value of y, the value of x must be multiplied by $\frac{1}{a}$ to obtain the same value of y. This has the effect of stretching the curve by a scale factor of $\frac{1}{a}$ in the x direction. (Of course, if a is greater than 1, then the graph will be compressed rather than stretched).

In general:

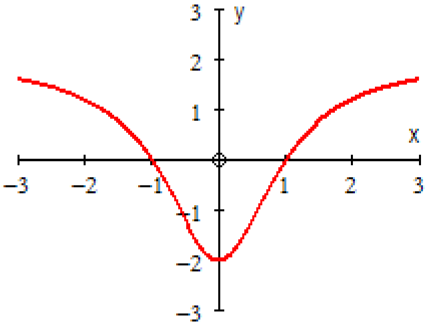
> For any function $\mathrm{f}(x)$, and any positive value of a, the graph of $y = \mathrm{f}(ax)$ can be obtained from the graph of $y = \mathrm{f}(x)$ by a stretch of scale factor $\frac{1}{a}$ parallel to the x-axis.

Similarly, the graph of $y = \mathrm{f}\left(\frac{x}{a}\right)$ is obtained from the graph of $y = \mathrm{f}(x)$ by a stretch of scale factor a parallel to the x-axis. So, just as replacing y with $\frac{y}{a}$ results in a stretch of scale factor a parallel to the y-axis, replacing x with $\frac{x}{a}$ results in a stretch of scale factor a parallel to the x-axis.

The diagram below shows a graph $y = \mathrm{f}(x)$ in red (in this case $\mathrm{f}(x) = x^3 - 4x$), the graph $y = \mathrm{f}(2x)$ in blue, and the graph $y = \mathrm{f}(\frac{1}{3}x)$ in purple.



Example 2
The diagram below shows the graph of $y = \mathrm{f}(x)$.



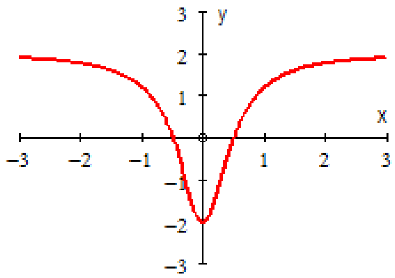
The graph cuts the x-axis at (1, 0) and (-1, 0) and the y-axis at (0, -2).

Sketch the graphs of:

(i) $y = \mathrm{f}(2x)$
(ii) $y = 3\mathrm{f}(x)$
(iii) $y = \mathrm{f}\left(\frac{1}{3}x\right)$
(iv) $y = \frac{1}{4}\mathrm{f}(x)$

giving the coordinates of the points where the graph crosses the coordinate axes in each case.

Solution
(i)

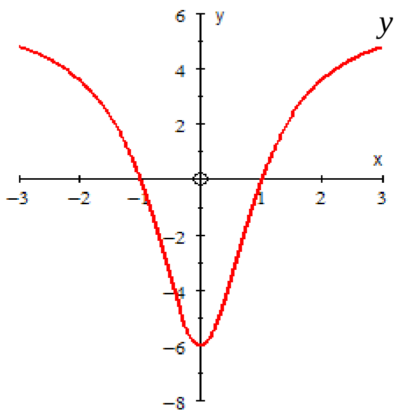


$y = \mathrm{f}(2x)$

$y = \mathrm{f}(2x)$ represents a one-way stretch scale factor $\frac{1}{2}$ parallel to the x-axis.

The graph cuts the x-axis at $\left(\frac{1}{2}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$ and the y-axis at (0, -2).

(ii)

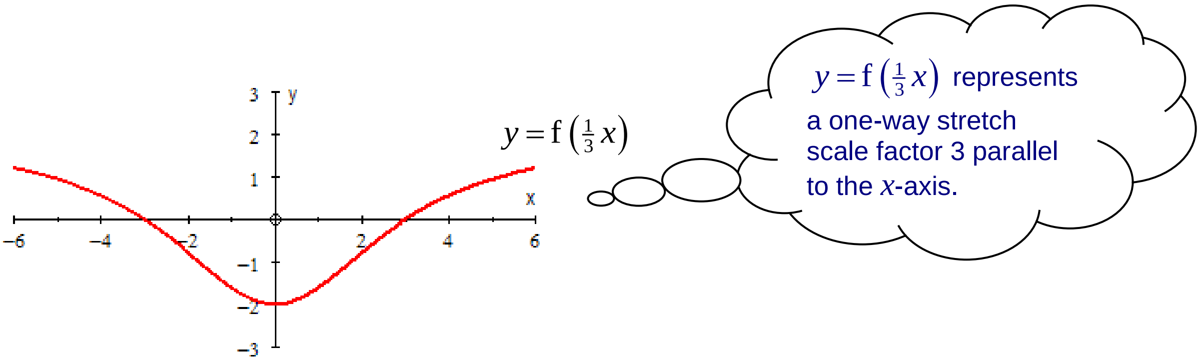


$y = 3\mathrm{f}(x)$

$y = 3\mathrm{f}(x)$ represents a one-way stretch scale factor 3 parallel to the y-axis.

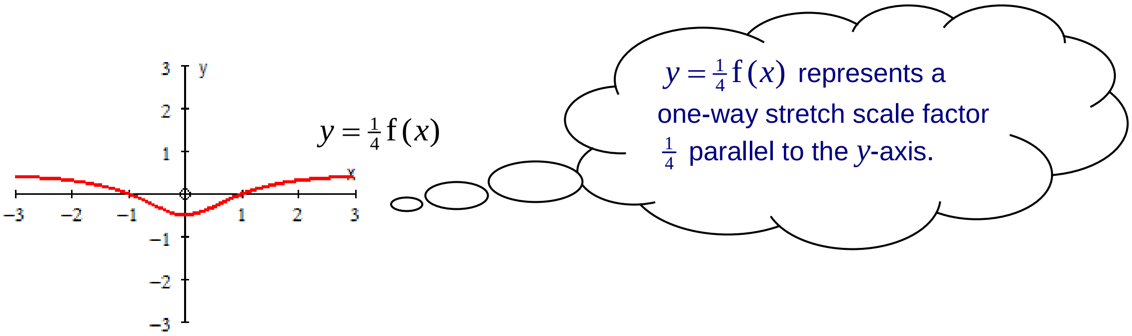
The graph cuts the x-axis at (1, 0) and (-1, 0) and the y-axis at (0, -6).

(iii)



The graph cuts the x-axis at (3, 0) and (-3, 0) and the y-axis at (0, -2).

(iv)



The graph cuts the x-axis at (1, 0) and (-1, 0) and the y-axis at $\left(0, -\tfrac{1}{2}\right)$.

Reflections

Use the ***Explore: Transformations*** resource to investigate graphs of the form $y = -\mathrm{f}(x)$ and $y = \mathrm{f}(-x)$

In fact these are just special cases of the one-way stretches you already know about. The equation $y = -\mathrm{f}(x)$ represents a stretch with scale factor -1 parallel to the y axis, which is the same as reflection in the x axis. The equation $y = \mathrm{f}(-x)$ represents a stretch with scale factor -1 parallel to the x axis, which is the same as reflection in the y axis.

- $y = -\mathrm{f}(x)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is reflected in the x axis
- $y = \mathrm{f}(-x)$ is the equation of the graph obtained when the graph of $y = \mathrm{f}(x)$ is reflected in the y axis

Example 3

Each of the following transformations is applied to the graph of the quadratic function $\mathrm{f}(x) = x^2 + 2x - 1$. Find the equation of the new curve in each case.

(i) Horizontal translation 3 units to the left

(ii) Stretch, scale factor 2, parallel to the y-axis

(iii) Reflection in the y-axis

Solution

(i) $y = \mathrm{f}(x+3)$

$= (x+3)^2 + 2(x+3) - 1$

$= x^2 + 6x + 9 + 2x + 6 - 1$

$= x^2 + 8x + 8$

(ii) $y = 2\mathrm{f}(x)$

$= 2(x^2 + 2x - 1)$

$= 2x^2 + 4x - 2$

(iii) $\mathrm{f}(-x) = (-x)^2 + 2(-x) - 1$

$= x^2 - 2x - 1$

Transformations of trigonometric graphs

All the rules above can be applied to trigonometric graphs.

You can investigate the effect of translations and stretches of the graphs of trigonometric functions using the Explore resource ***Transformations of trigonometric graphs***.

Example 4

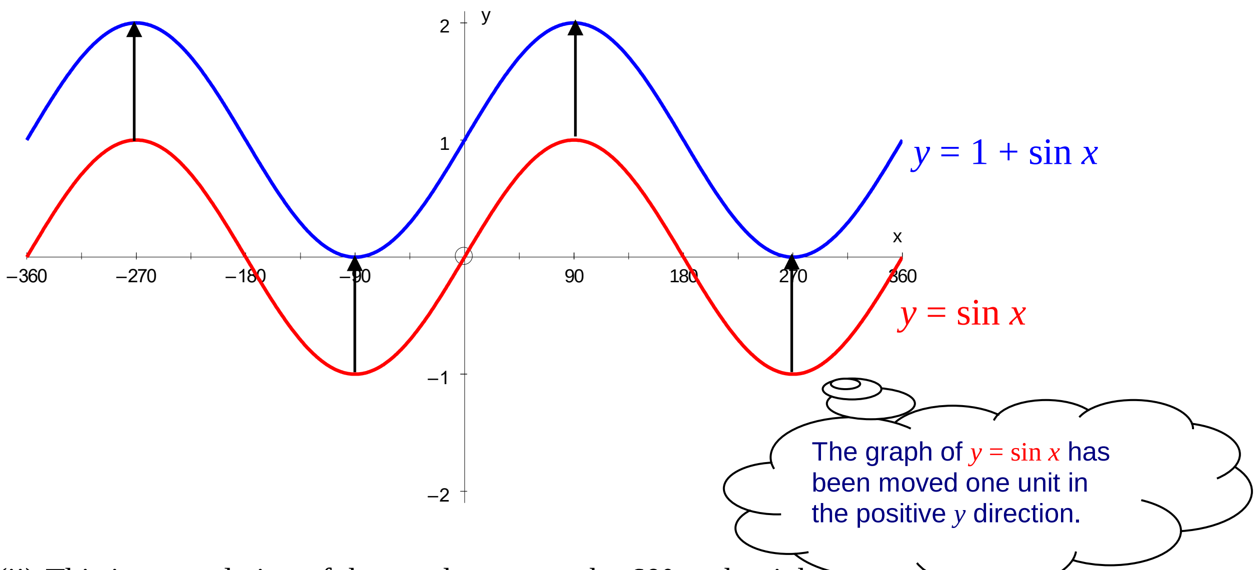
Sketch the graphs of

(i) $y = 1 + \sin x$

(ii) $y = \cos(x - 60°)$

Solution

(i) This is a translation of the graph $y = \sin x$ by 1 unit vertically upwards.



(ii) This is a translation of the graph $y = \cos x$ by 60° to the right.

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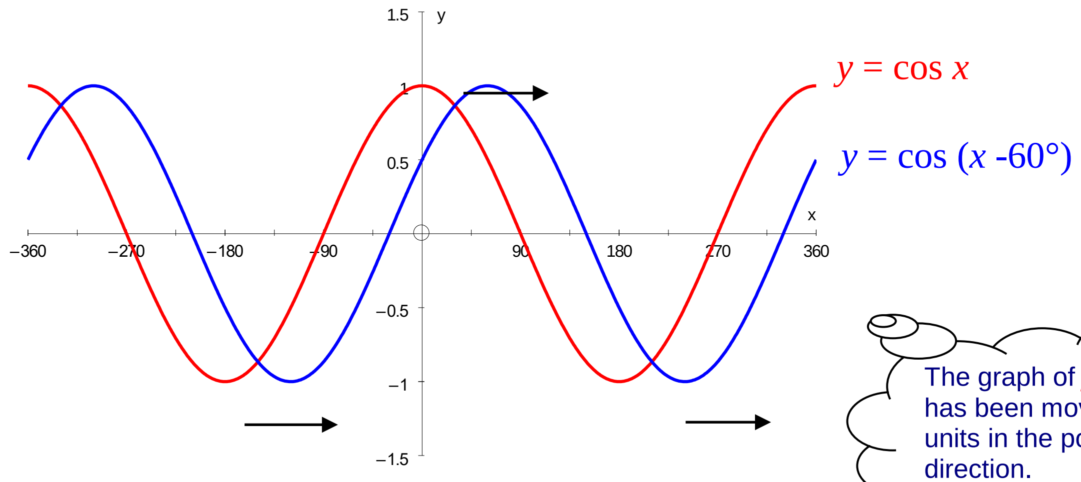


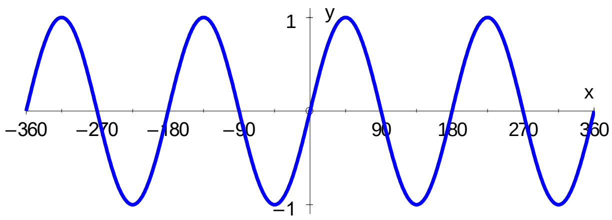
The graph of $y = \cos x$ has been moved 60 units in the positive x direction.

Horizontal stretches affect the **period** of a trigonometric graph. For example, the graph of $y = \cos 2x$ is obtained by stretching the graph of $y = \cos x$ by a factor of $\frac{1}{2}$ (i.e. compressing it). This graph therefore has a period half that of the period of the graph of $y = \cos x$.

Vertical stretches affect the amplitude of a trigonometric graph. For example, the graph of $y = 2 \sin x$ is obtained by stretching the graph of $y = \sin x$ by a factor of 2. The period of this graph is the same as that of $y = \sin x$, but its amplitude is doubled.

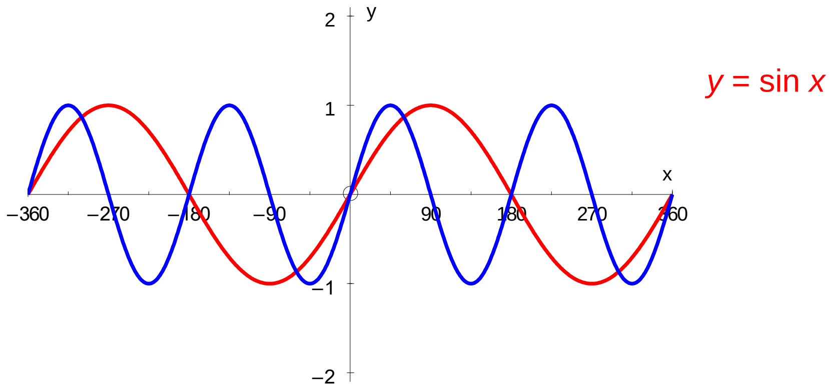
Example 4

Write down the equation of the following graph:



Solution

Compare the graph with that of $y = \sin x$:



The x co-ordinate of every point on the graph of $y = \sin x$ has been halved. Therefore the equation is $y = \sin 2x$.