Section 3: Further integration

Notes and Examples

These notes contain subsections on

- Integrating kx^n for negative and fractional n
- Applications of integration

Integrating kx^n for negative and fractional n

In Section 1 you saw that the integral of x^n, where n is a positive integer, is given by

$$\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c \qquad \text{where } c \text{ is an arbitrary constant}$$

In fact this formula is true not only when n is a positive integer, but for all real values of n, including negative numbers and fractions, except for $n = -1$.

The formula does not work for $n = -1$, since this would give a denominator of 0. There is a different way to integrate $\frac{1}{x}$, which is covered in later in A level Mathematics.

Example 1

Find the following indefinite integrals

(i) $\int \sqrt{x}\,\mathrm{d}x$

(ii) $\int \frac{1}{x^3}\,\mathrm{d}x$

(iii) $\int \left(\frac{2}{x^2} - \frac{3}{\sqrt{x}}\right)\mathrm{d}x$

Solution

(i) $\int \sqrt{x}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x$

$= \frac{2}{3}x^{\frac{3}{2}} + c$

$n = \frac{1}{2}$, so $n+1 = \frac{3}{2}$, so divide by $\frac{3}{2}$, i.e. multiply by $\frac{2}{3}$.

(ii) $\int \frac{1}{x^3}\,\mathrm{d}x = \int x^{-3}\,\mathrm{d}x$

$= -\frac{1}{2}x^{-2} + c$

$n = -3$, so $n+1 = -2$, so divide by -2.

(iii) $\displaystyle\int\left(\frac{2}{x^2}-\frac{3}{\sqrt{x}}\right)\mathrm{d}x=\int\left(2x^{-2}-3x^{-\frac{1}{2}}\right)\mathrm{d}x$

$=-2x^{-1}-3\times 2x^{\frac{1}{2}}+c$

$\displaystyle=-\frac{2}{x}-6\sqrt{x}+c$

$n=-2$, so $n+1=-1$, so divide by -1.

$n=-\frac{1}{2}$, so $n+1=\frac{1}{2}$, so divide by $\frac{1}{2}$, i.e. multiply by 2.

Example 2
Find the following definite integrals.

(i) $\displaystyle\int_1^2\left(\frac{4x-1}{x^4}\right)\mathrm{d}x$

(ii) $\displaystyle\int_1^4(3-x)\sqrt{x}\ \mathrm{d}x$

Solution

(i) $\displaystyle\int_1^2\left(\frac{4x-1}{x^4}\right)\mathrm{d}x=\int_1^2\left(4x^{-3}-x^{-4}\right)\mathrm{d}x$

$=\left[4\times-\frac{1}{2}x^{-2}+\frac{1}{3}x^{-3}\right]_1^2$

$\displaystyle=\left[-\frac{2}{x^2}+\frac{1}{3x^3}\right]_1^2$

$=\left(-\frac{1}{2}+\frac{1}{24}\right)-\left(-2+\frac{1}{3}\right)$

$=\frac{29}{24}$

Substitute $x = 2$ in first bracket and $x = 1$ in second bracket

(ii) $\displaystyle\int_1^4(3-x)\sqrt{x}\ \mathrm{d}x=\int_1^4\left(3x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)\mathrm{d}x$

$=\left[3\times\frac{2}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\right]_1^4$

$=\left[2x\sqrt{x}-\frac{2}{5}x^2\sqrt{x}\right]_1^4$

$=\left(2\times4\times2-\frac{2}{5}\times16\times2\right)-\left(2\times1\times1-\frac{2}{5}\times1\times1\right)$

$=\frac{8}{5}$

Substitute $x = 4$ in first bracket and $x = 1$ in second bracket

Applications of integration

Now that you can integrate a wider range of functions, you can also solve problems which involve integrating these functions, such as finding functions given their gradient function, and finding the area under a curve.

Example 3

The gradient function of a curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{x} - \frac{1}{\sqrt{x}}$$

and the curve passes through the point (4, 9)

Find the equation of the curve.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{x} - \frac{1}{\sqrt{x}} \Rightarrow y = \int\left(3\sqrt{x} - \frac{1}{\sqrt{x}}\right)\mathrm{d}x$$

Integrate to find an expression for y in terms of x

$$= \left(\int 3x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$= 3\times\tfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

$$= 2x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

When $x = 4$, $y = 9$ $\Rightarrow 9 = 2\times 8 - 2\times 2 + c$

Substitute the coordinates of the given point to find the value of the constant c.

$$\Rightarrow c = 9 - 16 + 4$$

$$\Rightarrow c = -3$$

The equation of the curve is $y = 2x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 3$

The next two examples are about finding the area under a curve.

Example 4

Find the area under the graph $y = 1 + \sqrt{x}$ between $x = 0$ and $x = 4$.

Solution

$$\text{Area under graph } = \int_0^4 \left(1 + \sqrt{x}\right)\,\mathrm{d}x$$

$$= \int_0^4 \left(1 + x^{\frac{1}{2}}\right)\,\mathrm{d}x$$

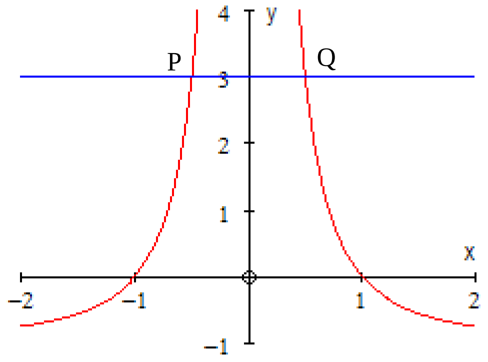
$$= \left[x + \tfrac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \left(4 + \tfrac{2}{3}\times 8\right) - 0$$

$$= \tfrac{28}{3}$$

Example 5

The diagram shows the graph of $y = \frac{1}{x^2} - 1$ and the line $y = 3$.



(i) Find the coordinates of points P and Q.

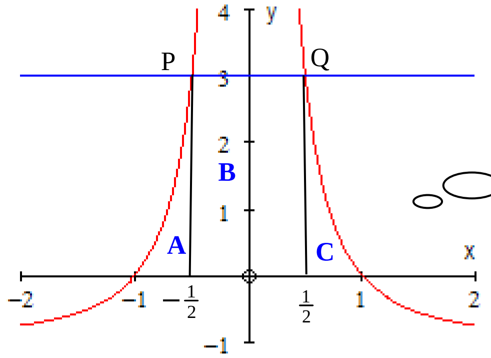
(ii) Find the area bounded by the curve, the line $y = 3$ and the x axis.

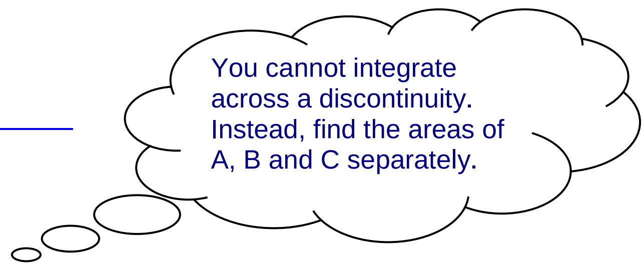
Solution

(i) At P and Q, $\frac{1}{x^2} - 1 = 3 \Rightarrow \frac{1}{x^2} = 4 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$.

The coordinates of P are $\left(-\frac{1}{2}, 3\right)$ and the coordinates of Q are $\left(\frac{1}{2}, 3\right)$.

(ii)





Area C is given by $\int_{\frac{1}{2}}^{1} \left(\frac{1}{x^2} - 1\right) dx = \left[-\frac{1}{x} - x\right]_{\frac{1}{2}}^{1}$

$= (-1-1) - \left(-2 - \frac{1}{2}\right)$

$= \frac{1}{2}$

By symmetry area A is also $\frac{1}{2}$.

Area B = $3 \times 1 = 3$

Total area = $\frac{1}{2} + \frac{1}{2} + 3 = 4$.