Section 1: Introduction to integration

These notes contain subsections on:
- **Reversing differentiation**
- **The rule for integrating x^n**
- **Indefinite integrals: formal notation**
- **Finding the arbitrary constant**

Reversing differentiation

Integration is the reverse of differentiation. If you are given an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$, and you want to find an expression for y, you need to use integration.
This is sometimes called solving a differential equation.

Remember that when you integrate, you must always add an arbitrary constant (see the textbook for the explanation of this).

Example 1 shows how you can integrate a function by thinking about what function you would need to differentiate to obtain the given function.

Example 1
Find y as a function of x for each of the following.

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = x^3$ (ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = x^6$ (iii) $\frac{\mathrm{d}y}{\mathrm{d}x} = x$

(iv) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^2$ (v) $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^4$ (vi) $\frac{\mathrm{d}y}{\mathrm{d}x} = 5$

Solution

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 \Rightarrow y = \frac{1}{4}x^4 + c$

The derivative of x^4 is $4x^3$.
So integrating $4x^3$ gives x^4.
Therefore integrating x^3 gives $\frac{1}{4}x^4$.

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = x^6 \Rightarrow y = \frac{1}{7}x^7 + c$

The derivative of x^7 is $7x^6$.
So integrating $7x^6$ gives x^7.
Therefore integrating x^6 gives $\frac{1}{7}x^7$.

(iii) $\frac{\mathrm{d}y}{\mathrm{d}x} = x \Rightarrow y = \frac{1}{2}x^2 + c$

The derivative of x^2 is $2x$.
So integrating $2x$ gives x^2.
Therefore integrating x gives $\frac{1}{2}x^2$.

(iv) $\frac{dy}{dx} = 2x^2 \Rightarrow y = \frac{2}{3}x^3 + c$

The derivative of x^3 is $3x^2$.
So integrating $3x^2$ gives x^3.
Therefore integrating x^2 gives $\frac{1}{3}x^3$,
and integrating $2x^2$ gives $\frac{2}{3}x^3$.

(v) $\frac{dy}{dx} = 3x^4 \Rightarrow y = \frac{3}{5}x^5 + c$

The derivative of x^5 is $5x^4$.
So integrating $5x^4$ gives x^5.
Therefore integrating x^4 gives $\frac{1}{5}x^5$,
and integrating $3x^4$ gives $\frac{3}{5}x^5$.

(vi) $\frac{dy}{dx} = 5 \Rightarrow y = 5x + c$

The derivative of x is 1.
So integrating 1 gives x.
Therefore integrating 5 gives $5x$.

The rule for integrating x^n

The method for integrating any polynomial function can be summed up as:

- Integrating x^n, where n is a positive integer, gives $\frac{x^{n+1}}{n+1}$
- Integrating kx^n, where n is a positive integer and k is a constant, gives $\frac{kx^{n+1}}{n+1}$
- You can integrate the sum of any number of such functions by simply integrating one term at a time.

Indefinite integration: formal notation

In Example 1 an expression for $\frac{dy}{dx}$ was given and used to find an expression for y.
So you would write:

$$\frac{dy}{dx} = 2x \Rightarrow y = x^2 + c.$$

Using the formal notation, you would write this as:

$$\int 2x \, dx = x^2 + c$$

You would read this as "the integral of $2x$ with respect to x"

The next example shows integration expressed using formal notation.

Example 2

Integrate each of the following functions.

(i) $x^3 + 3x + 2$

(ii) $4x^2 - 5x - 1$

(iii) $(x+3)(x-2)$

Remember the arbitrary constant

Solution

(i) $\int \left(x^3 + 3x + 2\right) dx = \frac{1}{4}x^4 + \frac{3}{2}x^2 + 2x + c$

(ii) $\int \left(4x^2 - 5x - 1\right) dx = \frac{4}{3}x^3 - \frac{5}{2}x^2 - x + c$

(iii) $\int (x+3)(x-2)\, dx = \int \left(x^2 + x - 6\right) dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + c$

Finding the arbitrary constant

If you are given additional information, you can find the value of the arbitrary constant by substituting the given information. This is sometimes called finding the particular solution of a differential equation. The next example shows how this is done.

Example 3

The gradient of a curve at any point (x, y) is given by $\frac{dy}{dx} = x^2(2x + 1)$.

The curve passes through the point (1, 5).
Find the equation of the curve.

Solution:

$\frac{dy}{dx} = x^2(2x + 1) = 2x^3 + x^2$

Just as with differentiating, you need to expand the brackets first

Integrating: $y = 2 \times \frac{1}{4}x^4 + \frac{1}{3}x^3 + c$

$= \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

When $x = 1, y = 5$ $\quad \Rightarrow 5 = \frac{1}{2} + \frac{1}{3} + c$

$\Rightarrow c = 5 - \frac{1}{2} - \frac{1}{3} = \frac{25}{6}$

Substitute the given values of x and y

So the equation of the curve is $y = \frac{1}{2}x^4 + \frac{1}{3}x^3 + \frac{25}{6}$