Section 2: Natural logarithms and exponentials

Notes and Examples

These notes contain subsections on:

Natural logarithms

You have already met logarithms section 1. In this section you are looking at natural logarithms, which are logarithms to base e, where e is a particular irrational number ($e = 2.71828\ldots$. to 5 decimal places). Working with natural logarithms is quite straightforward: they obey exactly the same rules as all other logarithms, so you already know quite a lot about them. Your calculator will be able to work out natural logarithms – you should have a button marked "ln" (note that this is ln not In; some people get confused about this!)

You may be wondering (quite reasonably) what is so special about this number e, and why logarithms to this particular base are so useful. In fact, this number has very many interesting properties, some of which you will learn about in this module and in others:

- Natural logarithms are related to the area under the curve $y = \frac{1}{x}$. You will find out more about this in A level Mathematics.
- One of the interesting characteristics of the exponential function e^x is that the gradient of the graph of $y = e^x$ is equal to the value of e^x at all points.
- In A level Mathematics you will learn to form and solve simple differential equations, which often involve modelling with exponential functions.
- If you go on to study Further Maths 'A' level, you will find that the number e^x can be written as the infinite series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ (try working out the sum of the first 10 or so terms of this series for $x = 1$ on your calculator, and see how close you get to the value of e).

But for now, all that is necessary is to accept that e, like π, is a special and useful number, and that this means that logarithms to base e are also particularly useful.

Solving equations using natural logarithms and exponentials

Most of the work in this section involves the same techniques as you used in section 1. You need to use the same rules of logarithms, applied to natural logarithms:

$$\ln a + \ln b = \ln ab$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

$$\ln a^n = n \ln a$$

Also remember that since the $\log_a a = 1$ for any value of a, then

$$\ln \mathrm{e} = 1$$

This means that you can use natural logarithms to solve equations involving exponentials, as in Example 1.

Example 1

Solve the equation $\mathrm{e}^{1-3x} = 5$.

Solution

Take natural logarithms of both sides: $\ln \mathrm{e}^{1-3x} = \ln 5$

Using the laws of logarithms: $(1-3x)\ln \mathrm{e} = \ln 5$

Since ln e = 1: $1 - 3x = \ln 5$

$$x = \frac{1-\ln 5}{3} = -0.203$$

Similarly, the relationship between exponentials and logarithms

$$a = \ln b \Leftrightarrow e^a = b$$

allows you to solve equations involving natural logarithms.

Example 2

Solve the equation $\ln(2x+1) = 3$.

Solution

$\ln(2x+1) = 3 \quad \Rightarrow 2x + 1 = \mathrm{e}^3$

$$\Rightarrow x = \frac{e^3 - 1}{2} = 9.54$$

In Examples 1 and 2 above, you are using the fact that the exponential function and the natural logarithm function are inverses of one another, in the same way that squaring and the square root function are inverses of one another. (You will learn more about functions and their inverses in chapter 3).

In Example 1 you are "undoing" an exponential function by using natural logarithms, and in Example 2 you are "undoing" a natural logarithm by using an exponential.

The inverse nature of these two functions can be summed up as follows:

$$\ln e^x = x$$
$$e^{\ln x} = x$$

Exponential functions as models

An exponential function is any function of the form a^x. The function e^x is an example of an exponential function, and is often called "the exponential function".

Many real life situations can be modelled by exponential functions. In section 1 you saw situations which can be modelled by functions like $y = c \times a^{kt}$ (exponential growth) or $y = c \times a^{-kt}$ (exponential decay). Often, functions are modelled using the exponential function e^x, giving rise to models of the form $y = ce^{kt}$ or $y = ce^{-kt}$.

You can solve problems involving exponential growth and decay using logarithms.

Example 3

The temperature T°C of a cup of coffee after t minutes is given by $T = 20 + 60e^{-0.1t}$.

(i) What is the initial temperature of the coffee?

(ii) What is the temperature of the coffee after 5 minutes?

(iii) After how long is the temperature of the coffee 25°C?

(iv) What is the temperature of the room?

Solution

(i) When $t = 0$, $T = 20 + 60 = 80$.
The initial temperature of the coffee is 80°C.

(ii) When $t = 5$, $T = 20 + 60e^{-0.5} = 56.4$
The temperature after 5 minutes is 56.4°.

(iii) When $T = 25$, $25 = 20 + 60e^{-0.1t}$

$$5 = 60e^{-0.1t}$$
$$e^{-0.1t} = \tfrac{1}{12}$$

Taking logarithms: $-0.1t = \ln \tfrac{1}{12}$

$$y = -10 \ln \tfrac{1}{12} = 10 \ln 12 = 24.8$$

It takes 24.8 minutes.

(iv) As t becomes very large, the temperature approaches a limiting value of 20°C, which is the temperature of the room.

The derivative of the exponential function

As mentioned earlier, one of the interesting characteristics of the function $y = \mathrm{e}^x$ is that its gradient is equal to the value of the y-coordinate at every point on the graph.

So $y = \mathrm{e}^x \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$.

If you replace x with kx, you are stretching the graph parallel to the x-axis with scale factor $\frac{1}{k}$. This has the effect of making the graph k times as steep.

So $y = \mathrm{e}^{kx} \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = k\mathrm{e}^{kx}$.

This result allows you to find the rate of change of an exponential function.