Section 3: Modelling curves

Notes and Examples

These notes contain subsections on:

- Modelling curves of the form $y = kx^n$
- Modelling curves of the form $y = ka^x$

Modelling curves of the form $y = kx^n$

When you collect data from an experiment, you may want to find a relationship between two variables, such as the speed of a moving object at a particular time, or temperature of an object and its distance from a heat source. You may plot a graph of one variable against another to help find this relationship. However, unless the graph is a straight line, it may be difficult to see the relationship from the graph. The graphs of $y = x^2$, $y = x^3$ etc. look quite similar for $x \geq 0$, and as the experimental data may not be very accurate, it can be impossible to tell with any certainty what would be the best graph to model the data.

This is where logarithms can be very useful. If the relationship is of the form $y = kx^n$, then plotting $\log y$ against $\log x$ gives a straight line graph. (Note that you can use logs to any base for this – it is usual to use either logs to base 10, or natural logarithms, since both of these are easily found on a calculator).

$$y = kx^n$$
$$\Rightarrow \log y = \log kx^n$$
$$\Rightarrow \log y = \log k + \log x^n$$
$$\Rightarrow \log y = \log k + n \log x$$
$$\Rightarrow \log y = n \log x + \log k$$

This is of the form $y = mx + c$, so plotting $\log y$ against $\log x$ gives a straight line with gradient n and intercept $\log k$.

The value of n is therefore the gradient of the graph, and the value of k is found by taking the intercept of the graph and finding its inverse logarithm (i.e. $10^{\text{intercept}}$ if you are using logs to base 10, or $\mathrm{e}^{\text{intercept}}$ if you are using natural logarithms).

Example 1

The relationship between two variables x and y is believed to be of the form $y = kx^n$, where k and n are constants.

In an experiment, the following values of x and y are recorded.

x	1	2	3	4	5	6	7	8
y	1.98	1.39	1.16	1.01	0.91	0.82	0.75	0.72

Verify that the model $y = kx^n$ is appropriate and find the approximate values of the constants k and n.

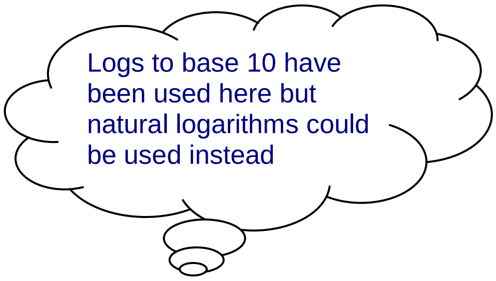
Solution

$$y = kx^n$$

Taking logarithms: $\log y = \log kx^n$

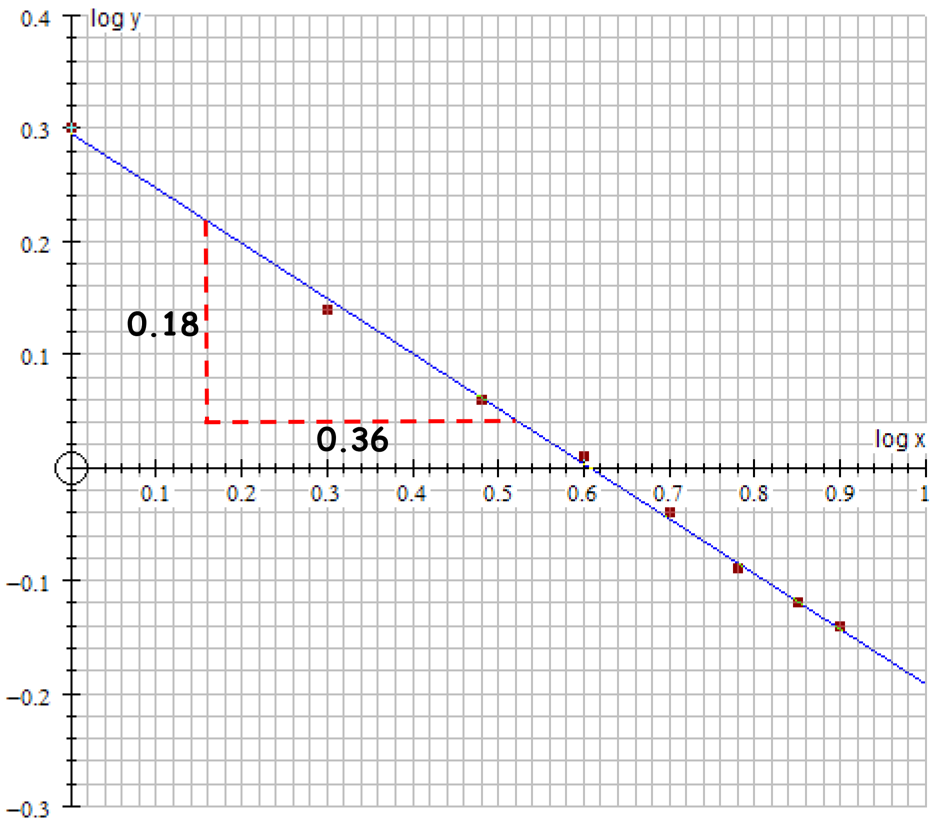
$$\log y = \log k + \log x^n$$

$$\log y = \log k + n \log x$$



If $\log y$ is plotted against $\log x$, this is the equation of a straight line graph with gradient n and intercept $\log k$.
Plot the values of $\log y$ against $\log x$:

x	1	2	3	4	5	6	7	8
$\log x$	0	0.30	0.48	0.60	0.70	0.78	0.85	0.90
y	1.98	1.39	1.16	1.02	0.91	0.82	0.75	0.72
$\log y$	0.30	0.14	0.06	0.01	-0.04	-0.09	-0.12	-0.14



Since the graph is approximately a straight line, the relationship $y = kx^n$ is an appropriate model.

Gradient = n = $-\dfrac{0.18}{0.36} = -0.5$

Intercept $= \log k = 0.3 \Rightarrow k = 10^{0.3} \approx 2$

The relationship is approximately $y = 2x^{-0.5} = \dfrac{2}{\sqrt{x}}$

Modelling curves of the form $y = ka^x$

Similarly, if the relationship is of the exponential form $y = ka^x$, then plotting $\log y$ against x gives a straight line graph.

$$y = ka^x$$
$$\Rightarrow \log y = \log ka^x$$
$$\Rightarrow \log y = \log k + \log a^x$$
$$\Rightarrow \log y = \log k + x \log a$$
$$\Rightarrow \log y = (\log a)x + \log k$$

This is of the form $y = mx + c$, so plotting $\log y$ against x gives a straight line with gradient $\log a$ and intercept $\log k$.

The value of a is found by taking the gradient of the graph and finding its inverse logarithm (i.e. 10^{gradient} if you are using logs to base 10 or e^{gradient} if you are using natural logs), and the value of k is found by taking the intercept of the graph and finding its inverse logarithm (i.e. $10^{\text{intercept}}$ if you are using logs to base 10 or $e^{\text{intercept}}$ if you are using natural logs).

Example 2

The relationship between two variables p and q is believed to be of the form $q = ab^p$, where a and b are constants.

In an experiment, the following values of p and q are recorded.

p	1.5	2.0	2.5	3.0	3.5	4.0
q	12	19	30	46	74	116

Verify that the model $q = ab^p$ is appropriate, and estimate the values of a and b.

Solution

$q = ab^p$

Taking logarithms: $\log q = \log ab^p$

$\log q = \log a + \log b^p$

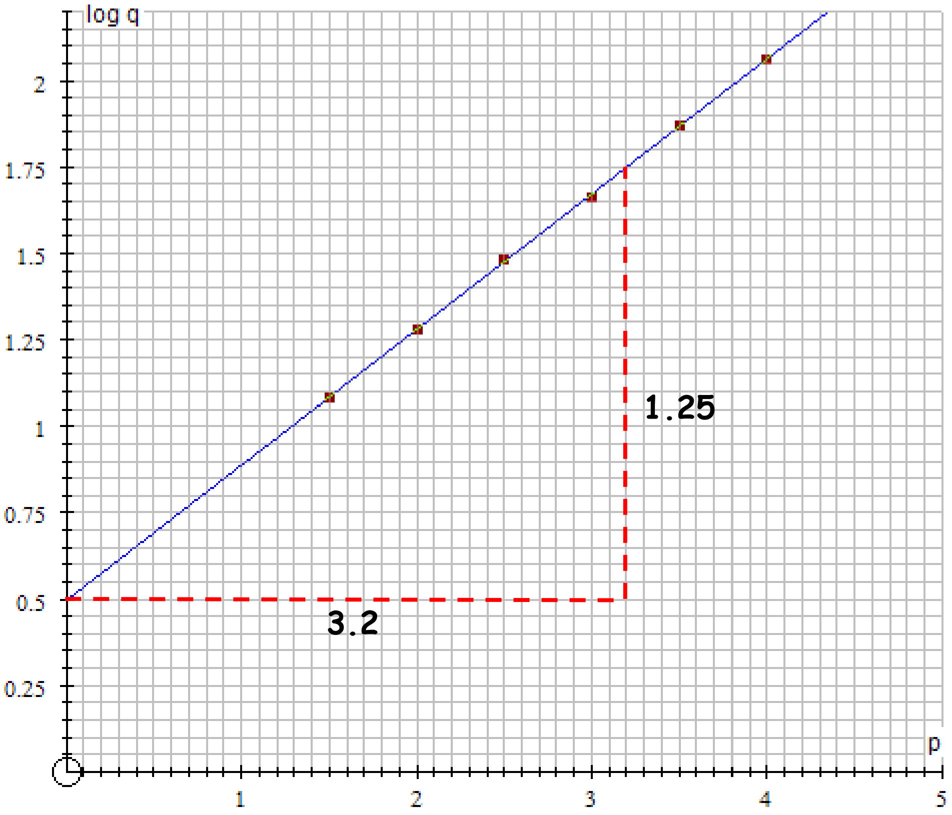
$\log q = \log a + p \log b$

If $\log q$ is plotted against p, this is the equation of a straight line with gradient $\log b$ and intercept $\log a$.

Again, this could be done using natural logs instead

Plot the values of $\log q$ against p:

p	1.5	2.0	2.5	3.0	3.5	4.0
q	12	19	30	46	74	116
$\log q$	1.08	1.28	1.48	1.66	1.87	2.06



Since the graph is approximately a straight line, the relationship $q = ab^p$ is an appropriate model.

Gradient = log b = $\frac{1.25}{3.2} \Rightarrow b = 10^{1.25/3.2} \approx 2.5$

Intercept = log a = 0.5 $\Rightarrow a = 10^{0.5} \approx 3.2$

The relationship is approximately $q = 3.2 \times 2.5^p$

You do not need to remember the details of what to plot and what to do with the gradient and intercept of the graph – all you need to do is to take logs of both sides of the suggested relationship and apply the laws of logarithms to obtain a relationship of the form $y = mx + c$, as shown above for each of the relationships $y = kx^n$ and $y = ka^x$. Once you have done this, you can see what you need to plot and how to find the values of the constants.

If the graph is not a straight line, then the suggested model is not an appropriate one (or perhaps the experimental results are not sufficiently accurate).