Section 4: More about differentiation

These notes contain sub-sections on:

Second derivatives

If you differentiate a derivative, you get the *second derivative*. If you start with an equation for y in terms of x, the first derivative is $\frac{\mathrm{d}y}{\mathrm{d}x}$ (you say: "dee y by dee x") and the second derivative is written $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ (you say: "dee two y by dee x squared")

The second derivative tells you about the rate of change of the derivative.

Example 1

Given that $y = 3x - x^3$, find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Solution

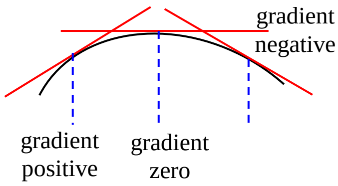
$$y = 3x - x^3$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3 - 3x^2$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6x$$

One important application of first and second derivatives is in the motion of a particle. You will learn more about this in your study of Mechanics.

The second derivative test for turning points

Maximum points

If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, the gradient function $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing.
At a maximum point, the gradient goes from + to 0 to –, in other words is decreasing.

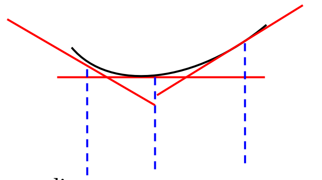


So $\frac{d^2y}{dx^2} < 0 \Rightarrow$ the turning point is a maximum.

Minimum points

If $\frac{d^2y}{dx^2} > 0$, the gradient function $\frac{dy}{dx}$ is increasing.

At a minimum point, the gradient goes from – to 0 to +, in other words is increasing.



So $\frac{d^2y}{dx^2} > 0 \Rightarrow$ the turning point is a minimum.

If the value of the second derivative is zero, this method cannot be used, and you must use the earlier method of looking at the sign of the gradient on either side of the point.

Example 2

Find the turning points of $y = 3x - x^3$ and determine their nature using the second derivative test.

Solution

$y = 3x - x^3$

$\Rightarrow \frac{dy}{dx} = 3 - 3x^2$

$\frac{dy}{dx} = 0$ when $3 - 3x^2 = 0$

$1 - x^2 = 0$

$(1-x)(1+x) = 0$

$\Rightarrow x = 1$ or -1

When $x = 1$, $y = 2$; when $x = -1$, $y = -2$

The stationary points are (1, 2) and (–1, –2)

$\frac{d^2y}{dx^2} = -6x$

When $x = 1$, $\frac{d^2y}{dx^2} = -6 < 0 \Rightarrow$ maximum

When $x = -1$, $\frac{d^2y}{dx^2} = 6 > 0 \Rightarrow$ minimum.

(1, 2) is a maximum point and (-1, -2) is a minimum point.

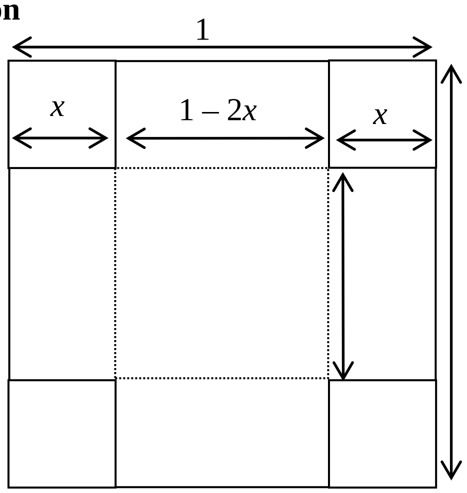
Maximum and minimum problems

One important immediate application of differentiation is to problems that involve maximising or minimising a variable quantity. You have already met the idea of finding maximum or minimum points on a graph using differentiation: now you will start to apply the same ideas to other types of problem.

Example 3

A rectangular sheet of metal of length 1 m and width 1 m has squares cut from each corner. The sides are then folded up to form an open topped box. Find the maximum possible volume of the box.

Solution



Step 1: Draw a diagram and use it to help you to formulate the problem mathematically. Call the side length of the squares cut out x. What are the length, width and depth of the box in terms of x?

Length = width $= 1 - 2x$

Depth = x

The volume V m^3 of the box is given by $V = x(1-2x)^2$.

Step 2: Find the maximum volume by differentiating. The maximum volume will occur when $\frac{dV}{dx} = 0$. Before differentiating, expand the brackets.

$$\begin{aligned} V &= x(1-2x)^2 \\ &= x(1-4x+4x^2) \\ &= x - 4x^2 + 4x^3 \end{aligned}$$

$$\frac{dV}{dx} = 1 - 8x + 12x^2$$

Now put $\frac{dV}{dx} = 0$, and solve for x:

$$1 - 8x + 12x^2 = 0$$
$$(2x-1)(6x-1) = 0$$
$$x = \tfrac{1}{2} \text{ or } x = \tfrac{1}{6}$$

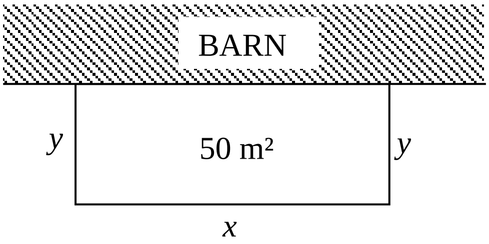
When $x = \frac{1}{2}$, $V = \frac{1}{2}\left(1 - 2\times\left(\frac{1}{2}\right)\right)^2 = 0$. This must be the minimum.

When $x = \frac{1}{6}$, $V = \frac{1}{6}\left(1 - 2\times\left(\frac{1}{6}\right)\right)^2 = \frac{4}{54} = \frac{2}{27}$. This must be the maximum.

So the maximum possible volume of the box is $\frac{2}{27}$ m^3.

Example 4

A farmer wants to make a pen for a goat, using the side of a barn as one side of the pen. He wants the pen to have an area of 50 m², but wants to use as little fencing as possible.



(i) Write down expressions for the area of the pen and the length of fencing required in terms of *x* and *y*.

(ii) Find an expression for the length of fencing required, *L*, in terms of *x* only.

(iii) Find $\frac{dL}{dx}$ and hence find the minimum length of fencing required, and show that this is a minimum.

Solution

(i) Area $= xy$

Length of fencing $= x + 2y$

(ii) $50 = xy \Rightarrow y = \frac{50}{x}$

$$L = x + 2y$$
$$= x + \frac{100}{x}$$

(ii) $\frac{dL}{dx} = 1 - \frac{100}{x^2}$

At minimum value of L, $\frac{dL}{dx} = 0 \Rightarrow 1 - \frac{100}{x^2} = 0$

$$\Rightarrow x^2 = 100$$
$$\Rightarrow x = 10$$

When $x = 10$, $L = 10 + \frac{100}{10} = 20$

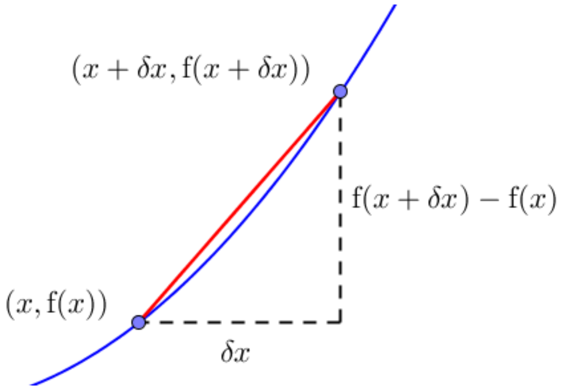
$$\frac{d^2L}{dx^2} = \frac{200}{x^3}$$

When $x = 10$, $\frac{d^2L}{dx^2} > 0$ so this is a minimum.

The minimum length of fencing required is 20m.

Differentiation from first principles

The notation $\frac{dy}{dx}$ for the derivative of a function comes from *differentiation from first principles*. To find the gradient function of a function f, you find the gradient of the chord which joins the point $(x, f(x))$ to another point $(x + \delta x, f(x + \delta x))$, where δx is very small.



When you have simplified this as much as possible, you then let $\delta x = 0$, and the chord becomes a tangent to the graph.

Example 5

Differentiate $y = x^3 - 2x$ from first principles.

Solution

$f(x) = x^3 - 2x$

Need to find the gradient of the chord joining the points $(x, f(x))$ and $(x+\delta x, f(x+\delta x))$

$$\text{Gradient } = \frac{f(x+\delta x) - f(x)}{(x+\delta x) - x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f(x+\delta x) = (x+\delta x)^3 - 2(x+\delta x) = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2x - 2\delta x$$

So $$f(x+\delta x) - f(x) = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2x - 2\delta x - (x^3 - 2x) = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2\delta x$$

$$\text{So gradient} = \frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - 2\delta x}{\delta x} = 3x^2 + 3x\delta x + (\delta x)^2 - 2$$

Now letting $\delta x \to 0$, $\frac{dy}{dx} = 3x^2 - 2$

Notice that this is the same answer you would obtain by applying the rules to find $\frac{dy}{dx}$.
By differentiating from first principles you can see that the rules do indeed give the correct gradient functions (derivatives).