Section 1: Points and straight lines

Notes and Examples

These notes contain sub-sections on:

- Gradients, distances and mid-points
- The equation of a straight line
- The intersection of two lines

Gradients, distances and mid-points

The ***Explore resource Points*** looks at how you can find the midpoint of two points, the distance between two points and the gradient of a line joining two points.

You will have met gradients before at GCSE. Remember that lines which go "downhill" have negative gradients.

To find the gradient of a straight line between two points (x_1, y_1) and (x_2, y_2), use the formula

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If two lines are parallel, they have the same gradient.
If two lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$

Example 1
P is the point (-3, 7). Q is the point (5, 1).
Calculate
(i) the gradient of PQ
(ii) the gradient of a line parallel to PQ
(iii) the gradient of a line perpendicular to PQ.

Solution
(i) Choose P as (x_1, y_1) and Q as (x_2, y_2).

or vice versa: it will still give the same answer (***WHY***?)

$$\text{Gradient of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-7}{5-(-3)} = \frac{-6}{8} = -\frac{3}{4}$$

Notes:
(1) Draw a sketch and check that your answer is sensible (e.g. has negative gradient).
(2) Check that you get the same result when you choose Q as (x_1, y_1) and P as (x_2, y_2).

(ii) When two lines are parallel their gradients are equal. $(m_1 = m_2)$

So the gradient of the line parallel to PQ is also $-\frac{3}{4}$.

(iii) When two lines are perpendicular $m_1 m_2 = -1$.

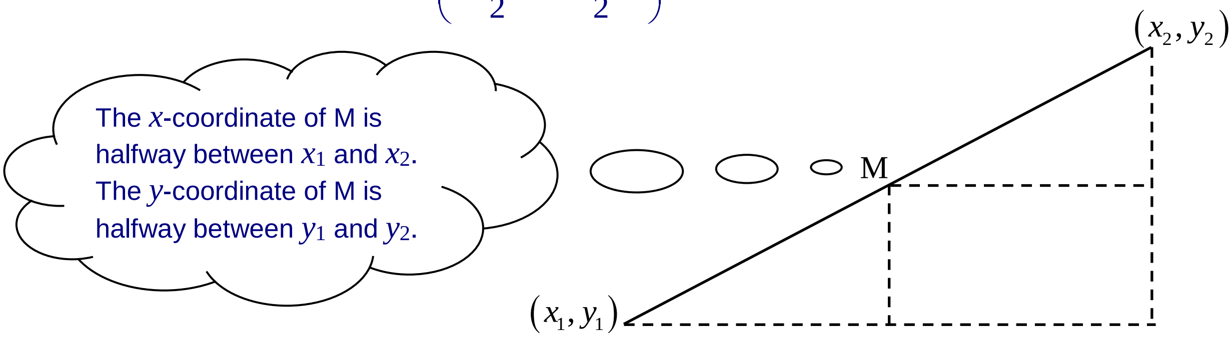
So $-\frac{3}{4}m_2 = -1$

$\Rightarrow m_2 = \frac{4}{3}$

The gradient of a line perpendicular to PQ is $\frac{4}{3}$.

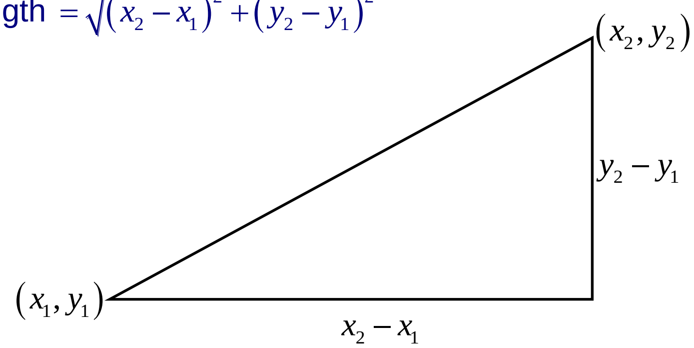
The midpoint of a line joining two points (x_1, y_1) and (x_2, y_2) is given by

$$\text{Midpoint } = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

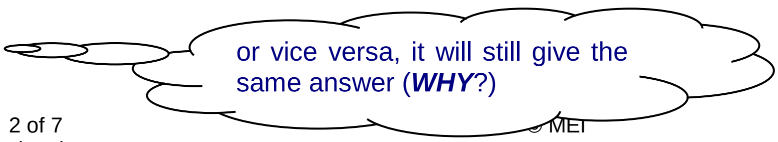


The length of a line joining two points (x_1, y_1) and (x_2, y_2) can be found using Pythagoras' Theorem.

$$\text{Length } = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 2
A is the point (2, -6). B is the point (-3, 4).
Calculate
(i) the midpoint of AB
(ii) the length of AB.
Choose A as (x_1, y_1) and B as (x_2, y_2).



Solution

(i) Midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

i.e. $\left(\frac{2+(-3)}{2}, \frac{-6+4}{2}\right)$

$= \left(\frac{-1}{2}, -1\right)$

(ii) The distance AB is given by

$$\begin{aligned} d &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ &= \sqrt{(2-(-3))^2+((-6)-4)^2} \\ &= \sqrt{(5)^2+(-10)^2} \\ &= \sqrt{25+100} \\ &= \sqrt{125} \end{aligned}$$

Note: The answer is often left like this if the square root is not exact. However since 125 = 25×5 then $\sqrt{125} = \sqrt{25}\sqrt{5} = 5\sqrt{5}$ is perhaps a simpler form.

For further practice in examples like the one above, try the ***Points skill pack***.

The equation of a straight line

The equation of a straight line is often written in the form $y = mx + c$, where m is the gradient and c is the intercept with the y-axis.

Example 3

Find (i) the gradient and (ii) the y-intercept of the following straight-line equations.

(a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

Solution

(a) Rearrange the equation into the form $y = mx + c$.

$5y = 7x - 3$ becomes $y = \frac{7}{5}x - \frac{3}{5}$

so $m = \frac{7}{5}$ and $c = -\frac{3}{5}$

Note the minus sign

(i) The gradient is $\frac{7}{5}$

(ii) The y-intercept is $-\frac{3}{5}$.

(b) Rearrange the equation into the form $y = mx + c$.

$3x + 8y - 7 = 0$ becomes $8y = -3x + 7$

giving $y = -\frac{3}{8}x + \frac{7}{8}$

so $m = -\frac{3}{8}$ and $c = \frac{7}{8}$

Note the minus sign

(i) The gradient is $-\frac{3}{8}$

(ii) The y-intercept is $\frac{7}{8}$.

Sometimes you may need to sketch the graph of a line. A sketch is a simple diagram showing the line in relation to the origin. It should also show the coordinates of the points where it cuts one or both axes.

You can explore straight line graphs using the Explore resources ***Straight lines*** and ***Parallel and perpendicular lines***. You may also find the Mathcentre video ***Equations of a straight line*** and ***Linear functions and graphs*** useful.

Example 4

Sketch the lines (a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

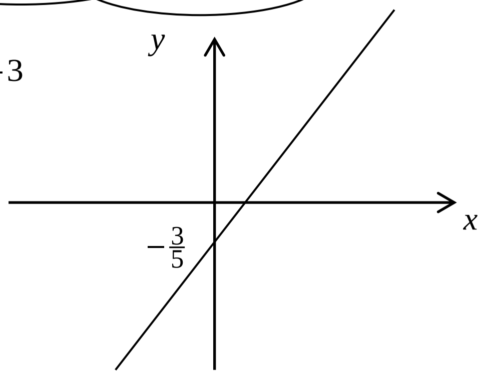
Solution

(a) From Example 3 you know that $5y = 7x - 3$ has gradient $\frac{7}{5}$ and y-intercept $-\frac{3}{5}$.

The gradient is positive, so the line slopes upwards from left to right. It's also greater than 1 and so steeper than 45 degrees.

This means the line goes through (0, -0.6) which is below the origin.

Sketch of $5y = 7x - 3$

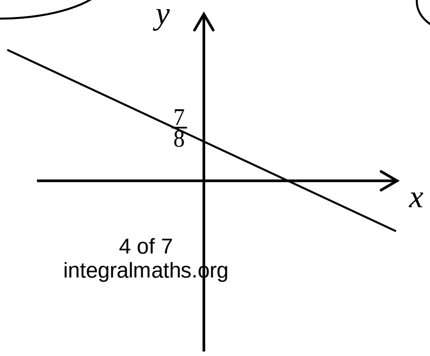


(b) From Example 3 you know that $3x + 8y - 7 = 0$ has gradient $-\frac{3}{8}$ and y-intercept $\frac{7}{8}$.

The gradient is negative, so the line slopes downwards from left to right. It is also and less than 1 and so less steep than 45 degrees.

This means the line goes through (0, 0.875) which is above the origin.

Sketch of $3x + 8y - 7 = 0$



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Sometimes you may need to find the equation of a line given certain information about it. If you are given the gradient and intercept, this is easy: you can simply use the form $y = mx + c$. However, more often you will be given the information in a different form, such as the gradient of the line and the coordinates of one point on the line (as in Example 5) or just the coordinates of two points on the line (as in Example 6).

In such cases you can use the alternative form of the equation of a straight line. For a line with gradient m passing through the point (x_1, y_1), the equation of the line is given by

$$y - y_1 = m(x - x_1).$$

Example 5

(i) Find the equation of the line with gradient 2 and passing through (3, -1).

(ii) Find the equation of the line perpendicular to the line in (i) and passing through (3, -1).

Solution

(i) The equation of the line is $y - y_1 = m(x - x_1)$

> $m = 2$ and (x_1, y_1) is (3, -1)

$$\Rightarrow y - (-1) = 2(x - 3)$$
$$\Rightarrow y + 1 = 2x - 6$$
$$\Rightarrow y = 2x - 7$$

> You should check that the point (3, -1) satisfies your line. If it doesn't, you must have made a mistake!

(ii) For two perpendicular lines $m_1 m_2 = -1$, so the gradient of the new line is $-\frac{1}{2}$.

The equation of the line is $y - y_1 = m(x - x_1)$

> $m = -\frac{1}{2}$ and (x_1, y_1) is (3, -1)

$$\Rightarrow y - (-1) = -\tfrac{1}{2}(x - 3)$$
$$\Rightarrow -2y - 2 = x - 3$$
$$\Rightarrow -2y = x - 1$$
$$\Rightarrow y = -\tfrac{1}{2}x + \tfrac{1}{2}$$

> The final form of the equation can be written in various different ways:
> e.g. $2y = -x + 1$ (This form has no fractions.)
> e.g. $2y + x = 1$ (This has no fractions and avoids having a negative sign at the start of the right hand side.)

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You can practice finding equations of lines using the ***Straight lines skill pack***. Also look at the ***Parallel and perpendicular lines skill pack***.

In the next example, you are given the coordinates of two points on the line.

Example 6
P is the point (3, 8). Q is the point (-1, 5).
Find the equation of PQ.

Solution
Choose P as (x_1, y_1) and Q as (x_2, y_2).

One method is to find the gradient and then use this value and one of the points in $y - y_1 = m(x - x_1)$

$$\text{Gradient of PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-8}{-1-3} = \frac{-3}{-4} = \frac{3}{4}$$

Now use

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 8 = \tfrac{3}{4}(x-3)$$
$$\Rightarrow 4(y-8) = 3(x-3)$$
$$\Rightarrow 4y - 32 = 3x - 9$$
$$\Rightarrow 4y = 3x + 23$$

You should check that P and Q satisfy your line.

An alternative approach to the above examples is to put the formula for m into the straight line equation to obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

and then make the substitutions. This is equivalent to the first method, but does not involve calculating m separately first.

The intersection of two lines

The point of intersection of two lines is found by solving the equations of the lines simultaneously. This can be done in a variety of ways. When both equations are given in the form $y = \dots$ then equating the right hand sides is a good approach (see below). If both equations are not in this form, you can re-arrange them into this form first, then apply the same method. Alternatively, you can use the elimination method if the equations are in an appropriate form.

Example 7
Find the point of intersection of the lines $y = 3x - 2$ and $y = 5x - 8$.

Solution

$3x - 2 = 5x - 8$
$\Rightarrow -2 = 2x - 8$
$\Rightarrow 6 = 2x$
$\Rightarrow x = 3$

Substitute into one of the equations to find y

Substituting $x = 3$ into $y = 3x - 2$ gives $y = 3 \times 3 - 2 = 7$

The point of intersection is (3, 7)

Check that (3, 7) satisfies the second equation.