# Section 1: Modulus and argument

## Notes and Examples

These notes contain subsections on

- The modulus of a complex number
- The argument of a complex number
- Multiplying and dividing with the modulus-argument form

### The modulus of a complex number

You are familiar with describing a point in the plane using Cartesian coordinates. However, this is not the only way of describing the location of a point. One alternative is to give its distance from a fixed point (usually the origin) and a direction (in this case the angle between the line connecting the point to the origin, and the positive real axis).

This is a common method of describing locations in real life: you might say that a town is “50 miles north-west of London”, or when walking in open countryside your map might show you that you need to walk 2 miles on a bearing of 124°.
In mathematics there are some situations in which this method of describing points is more convenient than Cartesian coordinates.

The modulus of a complex number $z$ is the distance of the point representing $z$ from the origin on the Argand diagram. Notice that this definition also holds for real numbers on the number line: the modulus (or absolute value) of a real number is its distance on the number line from zero.

In the same way, $|z-w|$ (or $|w-z|$) is the distance of the point representing $z$ from the point representing $w$. This also holds for real numbers on the number line: the distance of a real number $x$ from a real number $y$ on the number line is $|x-y|$ (or $|y-x|$). For example, the distance between 2 and -3 on the number line is $|2-(-3)|=5$.

**Example 1**
Given that $z = 2 + 5\text{i}$ and $w = 3 - \text{i}$, find
(i) $|z|$
(ii) $|w|$
(iii) $|z-w|$

**Solution**
(i) $|z| = \sqrt{2^2+5^2} = \sqrt{29}$

(ii) $|w| = \sqrt{3^2+(-1)^2} = \sqrt{10}$

(iii) $z - w = 2 + 5\text{i} - (3 - \text{i}) = -1 + 6\text{i}$

$$|z - w| = \sqrt{(-1)^2 + 6^2} = \sqrt{37}$$

## The argument of a complex number

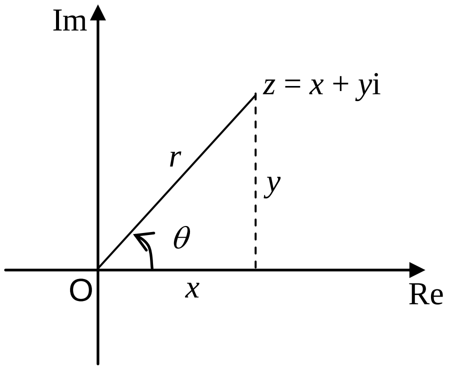
Finding the argument of a complex number involves using some knowledge of radians, and angles greater than 90°. You also need to know the values of the sine, cosine and tangent for common angles such as 30° $\left(\frac{\pi}{6}\text{ radians}\right)$, 45° $\left(\frac{\pi}{4}\text{ radians}\right)$ and 60° $\left(\frac{\pi}{3}\text{ radians}\right)$.

The diagram shows that the argument $\theta$ of the complex number $z = x + y\text{i}$ satisfies the equations

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

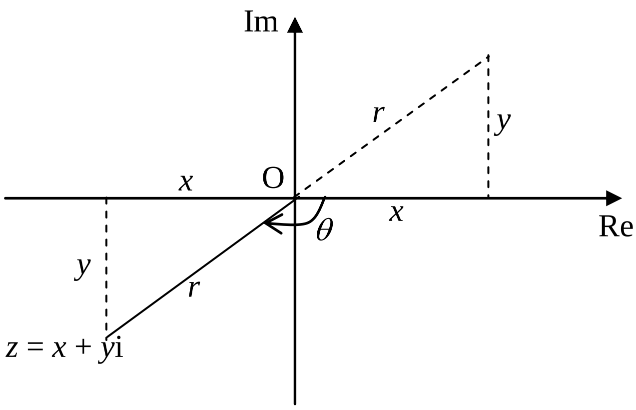
$$\tan\theta = \frac{y}{x}$$




In the diagram, the complex number $z$ lies in the first quadrant, since both $x$ and $y$ are positive. So tan $\theta$ is positive, and to find the value of $\theta$, you just need to find $\tan^{-1}\frac{y}{x}$.

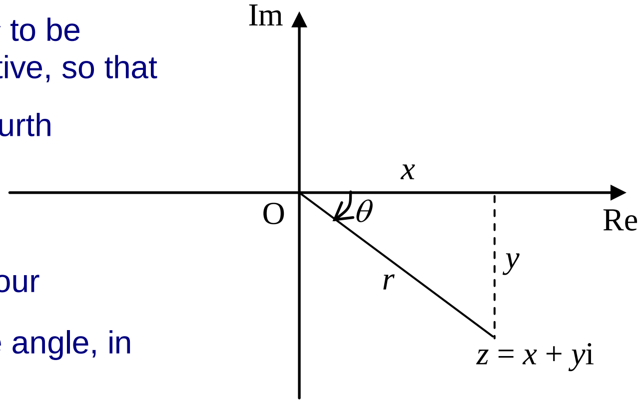
However, there is another possibility for which tan $\theta$ is positive. If both the real part and the imaginary part of $z$ are negative, $-\pi < \theta \le -\frac{\pi}{2}$. In this case $z$ is in the third quadrant.

However, as tan $\theta$ is positive, using your calculator to find $\tan^{-1}\frac{y}{x}$ will give you the corresponding angle in the first quadrant, (see diagram) where $x$ and $y$ are both positive. To find the correct argument, you need to subtract $\pi$.



Next we need to look at the cases where tan $\theta$ is negative.

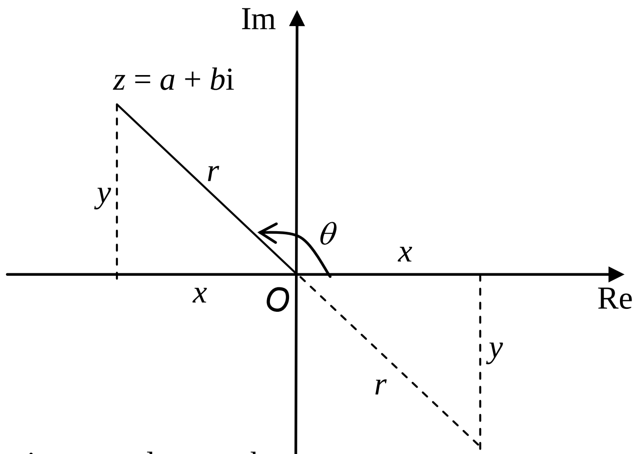
One possibility is for the real part of $z$ to be positive and the imaginary part negative, so that $-\frac{\pi}{2} \leq \theta < 0$. In this case, $z$ is in the fourth quadrant.




To find the value of $\theta$, find $\tan^{-1}\frac{y}{x}$. Your calculator should give you a negative angle, in the fourth quadrant as required.

The value of tan $\theta$ is also negative if the real part of $z$ is negative, but the imaginary part is positive, so that $\frac{\pi}{2} \leq \theta \leq \pi$. In this case $z$ is in the second quadrant.

Using your calculator to find $\tan^{-1}\frac{y}{x}$ will give you the corresponding angle in the fourth quadrant, with $x$ positive and $y$ negative (see diagram). To find the correct angle in the second quadrant, you need to add $\pi$.




**Example 2**

Find the modulus and argument of each of the following complex numbers.

(i) $z = 4 + 3i$ (ii) $z = -1 + i$ (iii) $z = -1 - \sqrt{3}i$

**Solution**

(i) $z = 4 + 3i$

$|z| = \sqrt{(3^2 + 4^2)} = 5$

Since $z$ lies in the first quadrant, arg $z = \tan^{-1}\frac{3}{4} \approx 0.644$

(ii) $z = -1 + i$

$|z| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

Since $z$ lies in the second quadrant, arg $z = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

(iii) $z = -1 - \sqrt{3}i$

Modulus = $\sqrt{1+3} = 2$

Since $z$ lies in the third quadrant, arg $z = \tan^{-1}\sqrt{3} - \pi = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

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You can use the modulus ($r$) and the argument ($\theta$) to write a complex number in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$. This is the modulus-argument form (sometimes called the polar form) of the complex number.

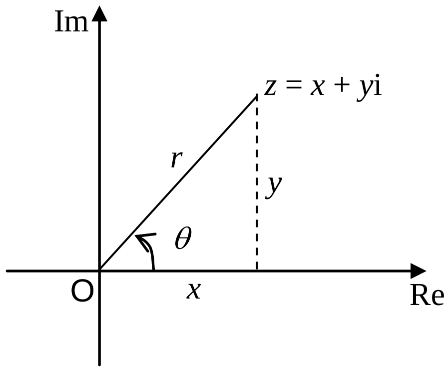
Make sure that you know the values of sin, cos and tan for the common angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ (i.e. 30°, 45°, 60°). When you come across these angles, you are expected to use the exact values.

Sometimes you may want to find a complex number if you are given its modulus and argument.

You can see from the diagram that

$x = r\cos\theta$

$y = r\sin\theta$




These relationships allow you to find the real and imaginary parts of a complex number with a given modulus and argument.

**Example 3**

Find the complex numbers with the given modulus and argument, in the form $x + \mathrm{i}y$.

(i) $|z| = 3, \quad \arg z = \frac{3\pi}{4}$

(ii) $|z| = 2, \quad \arg z = -\frac{\pi}{6}$

**Solution**

(i) $r = 3, \ \theta = \frac{3\pi}{4}$

$$x = 3\cos\frac{3\pi}{4} = 3 \times -\frac{1}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$$

$$y = 3\sin\frac{3\pi}{4} = 3 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$z = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}\mathrm{i}$$

(ii) $r = 2,\ \theta = -\dfrac{\pi}{6}$

$$x = 2\cos\left(-\frac{\pi}{6}\right) = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = 2\sin\left(-\frac{\pi}{6}\right) = 2\times -\frac{1}{2} = -1$$

$$z = \sqrt{3} - \mathrm{i}$$

## Multiplying and dividing using the modulus-argument form

Complex numbers in the modulus-argument form can be multiplied and divided easily by considering their moduli and arguments.

For two complex numbers w and z

$$|wz| = |w||z| \qquad \left|\frac{w}{z}\right| = \frac{|w|}{|z|}$$

$$\arg wz = \arg w + \arg z \qquad \arg\frac{w}{z} = \arg w - \arg z$$

In both the results above for the argument, some adjustment may be needed to ensure that the final argument is between -π and π, by adding or subtracting 2π. (This is illustrated in Example 4).

These results allow you to multiply and divide complex numbers in the modulus-argument form, quickly and easily.

These results can be interpreted geometrically using the Argand diagram:

- When $z$ is multiplied by $w$, the vector $z$ is enlarged by a scale factor $|w|$ and rotated through an angle of $\arg w$ anticlockwise about O
- When $z$ is divided by $w$, the vector $z$ is enlarged by a scale factor $\dfrac{1}{|w|}$ and rotated through an angle of $\arg w$ clockwise about O.

Example 4 shows how these relationships can be used.

**Example 4**

For the complex numbers $z = 3\left(\cos\dfrac{2\pi}{3} + \mathrm{i}\sin\dfrac{2\pi}{3}\right)$ and $w = 4\left(\cos\dfrac{\pi}{2} + \mathrm{i}\sin\dfrac{\pi}{2}\right)$, find, in modulus-argument form,

(i) $wz$ (ii) $\frac{w}{z}$.

Illustrate the points $w$, $z$, $wz$ and $\frac{w}{z}$ on an Argand diagram.

**Solution**

$|z| = 3, |w| = 4$

$\arg z = \frac{2\pi}{3}, \arg w = \frac{\pi}{2}$

(i) $|wz| = |w| \times |z| = 3 \times 4 = 12$

$$\arg wz = \arg w + \arg z = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$$

This cannot be the principal argument of $zw$ as it is not in the range $-\pi < \theta \le \pi$. The value must be adjusted by adding or subtracting multiples of $2\pi$.

Principal argument of $wz$ $= \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$

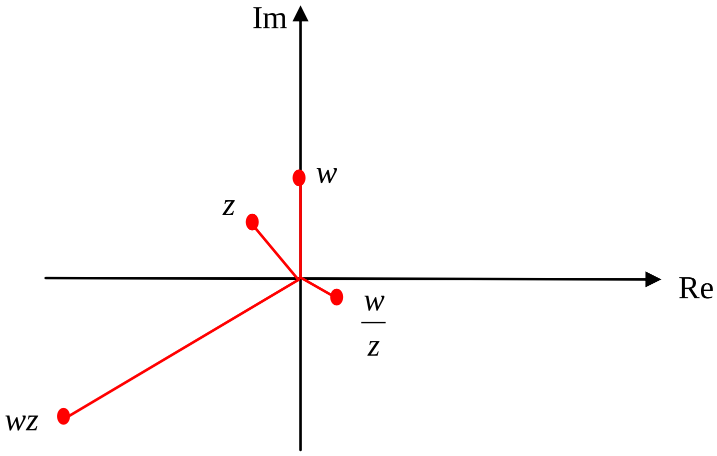
$$wz = 12\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$$

(ii) $\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{4}{3}$

$$\arg\frac{w}{z} = \arg w - \arg z = \frac{\pi}{2} - \frac{2\pi}{3} = -\frac{\pi}{6}$$

This time the argument is in the correct range so does not need adjusting.

$$\frac{w}{z} = \frac{4}{3}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$$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